Statistics 522, Problem Set 5 Solutions

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1. Suppose that $\{X_n, \mathcal{A}_n\}_{n\ge 0}$ is a martingale, and assume that $\{H_n, \mathcal{A}_n\}_{n\ge 0}$ is a predictable process: i.e. H_n is $\mathcal{A}_{n-1}$ measurable for each n. Then consider the new process $W_n \equiv (H\cdot X)_n \equiv \sum_{k=1}^n H_k(X_k - X_{k-1}) = \sum_{k=1}^n H_k \Delta X_k$.
(i) Show that if $\{H_n\}$ is a bounded (and predictable) process, then $\{W_n, \mathcal{A}_n\}_{n\ge 0}$ is also a martingale.
(ii) Show that the predictable variation process $\langle W\rangle_n$ of W_n is given by

$$\langle W\rangle_n = \langle H\cdot X\rangle_n = \sum_{k=1}^n H_k^2 E\left\{(\Delta X_k)^2 | \mathcal{A}_{k-1}\right\} + H_0^2 E(X_0^2).$$

and hence the sequence

$$L_n \equiv W_n^2 - \langle W\rangle_n \equiv \{(H\cdot X)_n\}^2 - \langle (H\cdot X)\rangle_n$$

is a 0−mean martingale with respect to the $\mathcal{A}_n$'s with $L_0 = H_0^2(X_0^2 - EX_0^2)$.

Solution: (i) First note that since $\sup_n |H_n| \le M < \infty$ for some M,

$$E|W_n| \le E\{\sum_{k=1}^n |H_k|\{|X_k| + |X_{k-1}|\} \le M\sum_{k=1}^n \{E|X_k| + E|X_{k-1}|\} < \infty.$$

Also, $\{W_n\}$ is adapted to $\{\mathcal{A}_n\}$ since $\{X_n\}$ is adapted to $\mathcal{A}_n$ and $\{H_n\}$ is predictable. Finally, for any n,

$$\begin{aligned} E\{W_{n+1}|\mathcal{A}_n\} &= E\{H_{n+1}(X_{n+1} - X_n) + \sum_{k=1}^n Y_k(X_k - X_{k-1})|\mathcal{A}_n\} \\ &= H_{n+1}E\{(X_{n+1} - X_n)|\mathcal{A}_n\} + W_n \quad \text{a.s.} \\ &= H_{n+1}\cdot 0 + W_n \quad \text{a.s.} \\ &= W_n. \end{aligned}$$

Thus $\{W_n, \mathcal{A}_n\}_{n\ge 0}$ is a martingale.
(ii) It follows from (i) that $\{W_n^2, \mathcal{A}_n\}_{n\ge 0}$ is a sub-martingale. For th proof of the given form for $\langle W\rangle_n$, see PfS course notes p. 373 or PfS (2017), page 365.

2. Suppose that $\{X_n, \mathcal{A}_n\}$ and $\{Y_n, \mathcal{A}_n\}$ are sub-martingales. Show that $X_n \vee Y_n = \max\{X_n, Y_n\}$ is a sub-martingale with respect to $\mathcal{A}_n$.

Solution: Let $Z_n \equiv X_n \vee Y_n$. Then

$$E(Z_{n+1}|\mathcal{A}_n) \;\ge\; E(X_{n+1}|\mathcal{A}_n) \ge X_n \quad \text{a.s.}$$

since $\{X_n\}$ is a sub-mg. Similarly $E(Z_{n+1}|\mathcal{A}_n) \geq Y_n$ almost surely. Combining these two statements yields $E(Z_{n+1}|\mathcal{A}_n) \geq \max\{X_n, Y_n\} = Z_n$ a.s.. Thus $\{Z_n, \mathcal{A}_n\}$ is a sub-martingale.

3. Suppose that X and Y are random variables on the probability space $(\Omega, \mathcal{A}, P)$ with $X \in L_2(P)$ and $Y \in L_2(P)$ (so that $XY \in L_1(P)$), and suppose that $\mathcal{D}$ is a sub sigma-field of $\mathcal{A}$. Show that

$$E\{XE(Y|\mathcal{D})\} = E\{E(X|\mathcal{D})Y\} = E\{E(X|\mathcal{D})E(Y|\mathcal{D})\}.$$

(With $\langle X, Y\rangle \equiv E(XY)$, this can be rewritten as

$$\langle X, E(Y|\mathcal{D})\rangle = \langle E(X|\mathcal{D}), Y\rangle = \langle E(X|\mathcal{D}), E(Y|\mathcal{D})\rangle,$$

and thus is the "self-adjointness" property of the conditional expectation operator.)

Solution: By computing conditionally on $\mathcal{D}$ we can write

$$\begin{aligned} E\{XE(Y|\mathcal{D})\} &= E\{E[XE(Y|\mathcal{D})|\mathcal{D}]\} \\ &= E\{E(Y|\mathcal{D})E[X|\mathcal{D}]\} \\ &= E\{E[YE(X|\mathcal{D})|\mathcal{D}]\} \\ &= E\{YE(X|\mathcal{D})\}. \end{aligned}$$

4. Exercise 12.4.1, page 313, PfS Course Notes, Chapter 12. (Exercise 12.4.1, page 309, PfS, 2017.)
Let T_1, $T_2, \ldots$ be (extended) stopping times; no ordering is assumed. Then:
(1) $T_1 + T_2$ is an extended stopping time if the $\mathcal{A}_t$'s are right-continuous.
(2) $A \in \mathcal{A}_{T_1}$ implies $A \cap [T_1 \leq T_2] \in \mathcal{A}_{T_2}$. Hint: $[T_1 \wedge t \leq T_2 \wedge t] \in \mathcal{A}_t$.
$[T_1 < T_2]$, $[T_1 = T_2]$, $[T_1 > T_2]$ are all in both $\mathcal{A}_{T_1}$ and $\mathcal{A}_{T_2}$.
(3) $T_1 \leq T_2$ implies $\mathcal{A}_{T_1} \subset \mathcal{A}_{T_2}$. Also $\mathcal{A}_{T_1} \cap [T_1 \leq T_2] \subset \mathcal{A}_{T_1 \wedge T_2} = \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$.
(4) If $T_n \searrow T_0$ and the $\mathcal{A}_t$'s are right continuous, then $\mathcal{A}_{T_0} = \cap_{n=1}^{\infty} \mathcal{A}_{T_n}$.

Solution: (5) Let T_1 and T_2 be (extended) stopping times. To see that $T_1 + T_2$ is an extended stopping time in the discrete time case (when $T_1, T_2 \in \mathbb{N} \cup \{\infty\}$), note that

$$[T_1 + T_2 \leq n] = \cup_{k=0}^{n} [T_1 = k] \cap [T_2 = n - k]$$

where $[T_1 = k] \in \mathcal{A}_k \subset \mathcal{A}_n$ and $T_2 = n - k] \in \mathcal{A}_{n-k} \subset \mathcal{A}_n$ for $k = 0, \ldots, n$. Hence $[T_1 + T_2 \leq n] \in \mathcal{A}_n$ and $T_1 + T_2$ is an (extended) stopping time.
For the continuous case we note (from the hint) that

$$[T_1 + T_2 \leq u] \quad = \quad \cap_{m=1}^{\infty} \{\cup_{a,b \in Q, a+b \leq u} [T_1 \leq a + 1/m] \cap [T_2 \leq b + 1/m]\}$$

where

$$[T_1 \le a + 1/m] \in \mathcal{A}_{a+1/m} \subset \mathcal{A}_{a+b+2/m} \subset \mathcal{A}_{u+2/m},$$
$$[T_2 \le b + 1/m] \in \mathcal{A}_{b+1/m} \subset \mathcal{A}_{a+b+2/m} \subset \mathcal{A}_{u+2/m},$$

for every $m \ge 1$. But this implies that $[T_1 + T_2 \le u] \in \mathcal{A}_{u+} = \mathcal{A}_u$.

(6) Suppose $A \in \mathcal{A}_{T_1}$. But then

$$\begin{aligned} & A \cap [T_1 \le T_2] \cap [T_2 \le t] \\ &= A \cap [T_1 \le t] \cap [T_1 \wedge t \le T_2 \wedge t] \cap [T_2 \le t] \\ &= \text{something in } \mathcal{A}_t \cap \text{something in } \mathcal{A}_t \cap \text{something in } \mathcal{A}_t \\ &\in \mathcal{A}_t. \end{aligned}$$

Hence $A \cap [T_1 \le T_2] \in \mathcal{A}_{T_2}$.
To see that $[T_1 < T_2]$, $[T_1 = T_2]$, and $[T_1 > T_2]$ are all in $\mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$, first note that

$$\begin{aligned} & [T_1 < T_2] \cap [T_2 \le t] \\ &= [T_1 < t] \cap [T_2 \le t] \cap [T_1 \wedge t < T_2 \wedge t] \\ &= \cup_{n=1}^{\infty} [T_1 \le t - 1/n] \cap [T_2 \le t] \cap [T_1 \wedge t < T_2 \wedge t] \\ &= \cup_{n=1}^{\infty} \text{something in } \mathcal{A}_{t-1/n} \cup \text{something in } \mathcal{A}_t \cap \text{something in } \mathcal{A}_t \\ &\in \mathcal{A}_t, \end{aligned}$$

so that $[T_1 < T_2] \in \mathcal{A}_{T_2}$. Then by taking $A = \Omega$ in the first part of the proof yields $[T_1 \le T_2] \in \mathcal{A}_{T_2}$, and hence also $[T_1 = T_2] = [T_1 < T_2]^c \cap [T_1 \le T_2] \in \mathcal{A}_{T_2}$. By symmetry we have $[T_1 = T_2] \in \mathcal{A}_{T_1}$ as well. And by symmetry again $[T_2 < T_1] \in \mathcal{A}_{T_1}$ and hence also $[T_2 \le T_1] = [T_2 < T_1] \cup [T_2 = T_1] \in \mathcal{A}_{T_1}$. This yields $[T_1 < T_2] = [T_2 \le T_1]^c \in \mathcal{A}_{T_1}$.

(7) If $A \in \mathcal{A}_{T_1}$, then $A \cap [T_2 \le t] = (A \cap [T_1 \le t]) \cap [T_2 \le t] \in \mathcal{A}_t$ for all $t \ge 0$ since $T_1 \le T_2$, and hence $A \in \mathcal{A}_{T_2}$.
Also, for $A \in \mathcal{A}_{T_1}$,

$$\begin{aligned} & A \cap [T_1 \le T_2] \cap [T_1 \wedge T_2 \le t] \\ &= A \cap [T_1 \le T_2] \cap ([T_1 \le t] \cup [T_2 \le t]) \\ &= (A \cap [T_1 \le T_2]) \cap [T_1 \le t] \quad \cup \quad (A \cap [T_1 \le T_2]) \cap [T_2 \le t] \\ &= \text{something in } \mathcal{A}_{T_1} \cap [T_1 \le t] \ \cup \ \text{something in } \mathcal{A}_{T_2} \cap [T_2 \le t] \\ &= \text{something in } \mathcal{A}_t \ \cup \ \text{something in } \mathcal{A}_t \\ &\in \mathcal{A}_t. \end{aligned}$$

for all $t \ge 0$, so $\mathcal{A}_{T_1} \cap [T_1 \le T_2] \subset \mathcal{A}_{T_1 \wedge T_2}$.
To see that $\mathcal{A}_{T_1 \wedge T_2} = \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$, suppose first that $A \in \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$. Then

$$A \cap [T_1 \wedge T_2 \le t] \;=\; [A \cap [T_1 \wedge T_2 \le t] \cap ([T_1 \le T_2] \cup [T_1 > T_2])$$

$$
\begin{aligned}
&= ((A \cap [T_1 \le T_2]) \cap [T_1 \le t]) \;\cup\; (A \cap [T_1 > T_2]) \cap [T_2 \le t] \\
&= \text{something in } \mathcal{A}_{T_1} \cap [T_1 \le t] \;\cup \text{something in } \mathcal{A}_{T_2} \cap [T_2 \le t] \\
&\qquad \text{by using } [T_1 \le T_2] \in \mathcal{A}_{T_1} \text{ and } [T_1 > T_2] \in \mathcal{A}_{T_2} \\
&\in \text{something in } \mathcal{A}_t \;\cup\; \text{something in } \mathcal{A}_t \\
&\in \mathcal{A}_t .
\end{aligned}
$$

For the reverse inclusion, suppose that $A \in \mathcal{A}_{T_1 \wedge T_2}$. But by the first part of the problem $T_1 \wedge T_2 \le T_j$ for $j = 1, 2$, and hence $\mathcal{A}_{T_1 \wedge T_2} \subset \mathcal{A}_{T_j}$ for $j = 1, 2$. Thus $A \in \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$.

(8) Since $T_0 \le T_n$ for all n, it follows from (7) that $\mathcal{A}_{T_0} \subset \mathcal{A}_{T_n}$ for all n and hence $\mathcal{A}_{T_0} \subset \cap_{n=1}^{\infty} \mathcal{A}_{T_n}$. To show the reverse inclusion, let $A \in \cap_{n=1}^{\infty} \mathcal{A}_{T_n}$. Since $A \cap [T_n \le t] \in \mathcal{A}_t$ for all $t \ge 0$ and $T_n \searrow T_0$, it follows that $A \cap [T_0 \le t] \in \mathcal{A}_t$ for all $t \ge 0$, and hence $A \in \mathcal{A}_{T_0}$.

5. Exercise 13.3.6, PfS Course Notes, page 359. [Exercise 13.3.6, PfS (2017), page 355.] Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a sub-martingale with $X_n \ge 0$. Let $r > 1$. Then $\{X_n^r\}$ is uniformly integrable if and only if $\{X_n^r\}$ is integrable.

   Solution: Uniform integrability implies integrability, so it remains only to prove the reverse implication. Suppose that $\{X_n^r\}$ is integrable. Then $\{X_n\}$ is uniformly integrable, and hence by the s-martingale convergence theorem 18.3.1, $X_n \to X_\infty \in L_1$ where $\{X_n, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a sub-mg; i.e. $E(X_\infty | \mathcal{A}_n) \ge X_n$ a.s. and

$$
E(X_\infty^r) = E(\liminf X_n^r) \le \liminf E(X_n^r) \le \sup_n E(X_n^r) < \infty
$$

   by Fatou's lemma and integrability of $\{X_n^r\}$. Hence by the conditional Jensen inequality,

$$
E(X_n^r) \le E\{E(X_\infty | \mathcal{A}_n)^r\} \le E\{E(X_\infty^r | \mathcal{A}_n)\} = E(X_\infty^r)
$$

   and it follows from Vitali's theorem that $\{X_n^r\}$ is uniformly integrable.

   Alternatively, by Doob's L_r-maximal inequality, since $\{X_n, \mathcal{A}_n\}$ is a sub-martingale,

$$
E\left\{ \left(\max_{1 \le k \le n} X_k \right)^r \right\} \le \left(\frac{r}{r-1} \right)^r E|X_n|^r ,
$$

   and hence, by the monotone convergence theorem,

$$
E[\sup_{1 \le k < \infty} X_k^r] \le \left(\frac{r}{r-1} \right)^r \sup_n E|X_n|^r < \infty .
$$

   Thus with $Y \equiv \sup_{1 \le k < \infty} X_k$, it follows that

$$
\sup_n E\left\{ X_n^r 1_{[X_n^r \ge \lambda]} \right\} \le E(Y^r 1_{[Y^r \ge \lambda]}) \to 0
$$

   as $\lambda \to \infty$; i.e. $\{X_n^r\}$ is uniformly integrable.