**Statistics 522, Problem Set 4 Solutions**

Wellner; 2/13/2020

1. Let $X_1, X_2, \ldots$ be independent rv's with each $X_k \ge 0$ and $EX_k = 1$. Let $M_n \equiv \prod_{k=1}^n X_k$ for $1 \le k \le n$ with $M_0 \equiv 1$. Then $\{M_n : \ n \ge 0\}$ is a martingale with all $E(M_n) = 1$.

   **Solution:** Let $\mathcal{A}_n \equiv \sigma[X_1, \ldots, X_n]$. Then $E|M_n| = E|\prod_{k=1}^n X_k| \le \prod_{k=1}^n E|X_k| < \infty$. That $E(M_n) = 1$ follows easily from independence of the $X_k$'s and the fact that $E(X_k) = 1$ for each $k$. Furthermore, since $X_{n+1}$ is independent of $\mathcal{A}_n$ and $E(X_{n+1}) = 1$,

$$
\begin{aligned}
E\{M_{n+1}|\mathcal{A}_n\} &= E\left\{X_{n+1}\prod_{k=1}^n X_k|\mathcal{A}_n\right\} = E\left\{X_{n+1}M_n|\mathcal{A}_n\right\} = M_n E\{X_{n+1}|\mathcal{A}_n\} \\
&= M_n \ \ \text{a.s.}
\end{aligned}
$$

   Thus $\{M_n, \mathcal{A}_n\}_{n\ge 1}$ is a mean 1 martingale.

2. Exercise 13.1.6, page 353, PfS Course Notes, Chapter 13. (Exercise 18.1.6, page 471, PfS, 2000.)
   Find the exponential martingale that corresponds to the martingale Poisson process $\mathbb{M}(t)$ in example 1.12. Then differentiate this twice with respect to $c$, set $c = 0$ each time, and obtain the two martingales given in the example.

   **Solution:** The martingale $\mathbb{M}(t)$ in example 1.12 is just $\mathbb{M}(t) = \mathbb{N}(t) - \lambda t$ where $\mathbb{N}$ is a Poisson process. The natural filtration $\{\mathcal{A}_t\}_{t\ge 0}$ is given by $\mathcal{A}_t \equiv \sigma[\mathbb{N}_(s) : \ \ s \le t]$. To get an exponential martingale, we define

$$
Y_c(t) \equiv \frac{\exp(c\mathbb{M}(t))}{E\exp(c\mathbb{M}(t))}. \tag{1}
$$

   Computing the denominator we find that

$$
\begin{aligned}
E\exp(c\mathbb{M}(t)) &= E\exp(c\mathbb{N}(t) - c\lambda t) = e^{-c\lambda t}E\exp(c\mathbb{N}(t)) = e^{-c\lambda t}\cdot e^{\lambda t(e^c-1)} \\
&= \exp(\lambda t(e^c - 1 - c)) \qquad (2)\\
&= \exp(\lambda(t-s)(e^c - 1 - c))\cdot\exp(\lambda s(e^c - 1 - c)) \\
&= E\exp(c(\mathbb{M}t) - \mathbb{M}(s))\cdot E\exp(c\mathbb{M}(s)). \qquad (3)
\end{aligned}
$$

using independence of $\mathbb{M}t) - \mathbb{M}(s)$ and $\mathbb{M}(s)$ in the last line, since, if $N \sim$ Poisson$(\gamma)$, the moment generating function is

$$Ee^{rN} = \sum_{k=0}^{\infty} e^{rk} e^{-\gamma} \frac{\gamma^k}{k!} = e^{-\gamma} \sum_{k=0}^{\infty} \frac{(e^r \gamma)^k}{k!} = e^{\gamma(e^r - 1)}.$$

Note that $E\{Y_c(t)\} = 1 < \infty$ for all $t > 0$. Furthermore, for $0 < s < t$, by using (3),

$$\begin{aligned}
E\{Y_c(t)|\mathcal{A}_s\} &= E\left\{\frac{\exp(c\mathbb{M}(t))}{E\exp(c\mathbb{M}(t))}|\mathcal{A}_s\right\} \\
&= E\left\{\frac{\exp(c(\mathbb{M}(t)-\mathbb{M}(s))+c\mathbb{M}(s))}{E\exp(c\mathbb{M}(t))}|\mathcal{A}_s\right\} \\
&= E\left\{\frac{\exp(c(\mathbb{M}(t)-\mathbb{M}(s)))}{E\exp(c(\mathbb{M}(t)-\mathbb{M}(s)))}\cdot\frac{\exp(c\mathbb{M}(s))}{E\exp(c\mathbb{M}(s))}|\mathcal{A}_s\right\} \\
&= \frac{\exp(c\mathbb{M}(s))}{E\exp(c\mathbb{M}(s))}\cdot E\left\{\frac{\exp(c(\mathbb{M}(t)-\mathbb{M}(s)))}{E\exp(c(\mathbb{M}(t)-\mathbb{M}(s)))}|\mathcal{A}_s\right\} \\
&= Y_c(s)\cdot 1 = Y_c(s) \quad \text{a.s.}
\end{aligned}$$

since $\mathbb{M}(t) - \mathbb{M}(s)$ is independent of $\mathcal{A}_s$. Thus $\{Y_c(t), \mathcal{A}_t\}_{t>0}$ is a mean 1 martingale.

Using (2) in (1) we find that

$$Y_c(t) = \exp(c\mathbb{M}(t) - \lambda t(e^c - 1 - c)).$$

Differentiating $Y_c$ with respect to $c$ yields

$$Y_c'(t) = Y_c(t) \cdot (\mathbb{M}(t) - \lambda t(e^c - 1)).$$

Since $Y_0(t) = 1$ and $e^0 = 1$, evaluating $Y_c'(t)$ at $c = 0$ yields just $\mathbb{M}(t)$. Differentiating with respect to $c$ again yields

$$Y_c''(t) = Y_c(t) \cdot (\mathbb{M}(t) - \lambda t(e^c - 1))^2 + Y_c(t) \cdot (-\lambda t e^c).$$

Evaluating this at $c = 0$ yields $\mathbb{M}^2(t) - \lambda t$. Thus both $\{\mathbb{M}(t), \mathcal{A}_t\}_{t\geq 0}$ and $\{\mathbb{M}^2(t) - \lambda t, \mathcal{A}_t\}_{t\geq 0}$ are martingales. This can also be verified by direct calculation.

3. Exercise 8.9.2, page 186: In the same context as Example 9.1, turn $\{S_k^2, \mathcal{A}_k\}_{1 \le k \le n}$ into a martingale by centering it appropriately.

**Solution:** If $X_k \sim (0, \sigma_k^2)$ are independent, then with $S_k \equiv X_1 + \cdots + X_k$ and $\mathcal{A}_k \equiv \sigma[X_1, \ldots, X_k]$, then I claim that $\{S_n^2 - \sum_{k=1}^n \sigma_k^2, \mathcal{A}_n\}_{n \ge 1}$ is a martingale. To see this we compute

$$E\left\{S_{n+1}^2 - \sum_{k=1}^{n+1} \sigma_k^2 | \mathcal{A}_n\right\} = E\left\{(X_{n+1} + S_n)^2 - \sigma_{n+1}^2 - \sum_{k=1}^{n} \sigma_k^2 | \mathcal{A}_n\right\}$$

$$= E\left\{X_{n+1}^2 - \sigma_{n+1}^2 | \mathcal{A}_n\right\} + S_n^2 - \sum_{k=1}^{n} \sigma_k^2$$

since the cross term $X_{n+1} S_n$ has zero conditional mean by independence of $X_{n+1}$ and $\mathcal{A}_n$, and since $E(X_{n+1}) = 0$

$$= 0 + S_n^2 - \sum_{k=1}^{n} \sigma_k^2 \quad \text{a.s..}$$

4. Exercise 8.10.1, page 189. (Exercise 8.11.1, page 249, PfS, 2000.) To complete the proof of the Hájek-Rényi inequality for martingales, show that $\{T_k, \mathcal{A}_k\}_{n \le k \le N}$ is a martingale and that $Var(T_N)$ is equal to the second factor on the right side of (b) on page 188, namely

$$\left\{\frac{\sum_{k=1}^n \sigma_k^2}{b_n^2} + \sum_{k=n+1}^{N} \frac{\sigma_k^2}{b_k^2}\right\}.$$

**Solution:** If $\{S_k, \mathcal{A}_k\}_{0 \le k \le N}$ is a $0-$mean martingale and $X_k \equiv S_k - S_{k-1}$ has $Var(X_k) = \sigma_k^2 < \infty$ for all $1 \le k \le N$, then with

$$T_k \equiv S_n / b_n + \sum_{j=n+1}^{k} X_j / b_j, \qquad \text{for } \; n \le k \le N,$$

$\{T_k, \mathcal{A}_k\}_{n \le k \le N}$ is a martingale. To see this, note that

$$\begin{aligned} E\{T_{k+1} | \mathcal{A}_k\} &= E\{S_n / b_n + \sum_{j=n+1}^{k+1} (X_j / b_j) | \mathcal{A}_k\} \\ &= S_n / b_n + \sum_{j=n+1}^{k} (X_j / b_j) + E\{(X_{k+1} / b_{k+1}) | \mathcal{A}_k\} \\ &= T_k \quad \text{a.s.} \end{aligned}$$

since $E(X_{k+1}|\mathcal{A}_k\} = E(S_{k+1}|\mathcal{A}_k\} - S_k = 0$ almost surely. Furthermore,

$$Var(T_N) \;=\; b_n^{-2}\sum_{k=1}^{n}\sigma_k^2 + \sum_{j=n+1}^{N}(\sigma_j^2/b_j^2)$$

since

$$\begin{aligned}
Var(T_N) &= Var(S_n/b_n) + 2b_n^{-1}Cov(S_n, \sum_{j=n+1}^{N}(X_j/b_j)) + Var(\sum_{j=n+1}^{N}X_j/b_j) \\
&= b_n^{-2}Var(S_n) + 0 + Var(\sum_{j=n+1}^{N}X_j/b_j) \\
&= b_n^{-2}\sum_{k=1}^{n}\sigma_k^2 + \sum_{j=n+1}^{N}\sigma_j^2/b_j^2
\end{aligned}$$

via the following additional calculations:

$$\begin{aligned}
Var(S_n) &= ES_n^2 = EE\{(S_{n-1}+X_n)^2|\mathcal{A}_{n-1}\} \\
&= E\{S_{n-1}^2 + 0 + E(X_n^2|\mathcal{A}_{n-1})\} \\
&= Var(S_{n-1}) + Var(X_n) = Var(S_{n-1}) + \sigma_n^2 \\
&= \cdots = \sum_{j=1}^{n}\sigma_j^2,
\end{aligned}$$

while, similarly,

$$\begin{aligned}
Var(\sum_{j=n+1}^{N}X_j/b_j) &= E\left\{\left(\sum_{n+1}^{N}X_j/b_j\right)^2\right\} \\
&= EE\left\{\left(\sum_{n+1}^{N-1}X_j/b_j\right)^2 + 2b_N^{-1}X_N\left(\sum_{n+1}^{N-1}X_j/b_j\right) + b_N^{-2}X_N^2|\mathcal{A}_N\right\} \\
&= Var\left(\sum_{j=n+1}^{N-1}X_j/b_j\right) + 0 + b_N^{-2}\sigma_N^2 \\
&= \cdots = \sum_{j=n+1}^{N}\sigma_j^2/b_j^2.
\end{aligned}$$

5. Let $Y_1, Y_2, \ldots$ be independent random variables, and suppose that $Y_k$ has either the density $p_k$ or $q_k$ with respect to some common dominating measure $\mu$. Let $X_k \equiv q_k(Y_k)/p_k(Y_k)$ for $k \geq 1$, let $P_k$ denote the probability measure on $\mathbb{R}$ corresponding to $p_k$, and let $P = \prod_{k=1}^{\infty} P_k$ denote the resulting product measure on $(\mathbb{R}^{\infty}, \mathcal{B}_{\infty})$.
(a) Relate the $X_k$'s above to Kakutani's martingale as in Example 13.1.14 (PfS, page 343).
(b) Relate the $X_k$'s above to the likelihood ratio martingale as in Example 13.1.13 (PfS, page 343).

**Solution:** (a) Note that under $P$ then $X_k = (q_k/p_k)(Y_k)$ satisfy

$$EX_k = \int_{\mathbb{R}} \frac{q_k(y_k)}{p_k(y_k)} dP_k(y_k) = \int_{\mathbb{R}} \frac{q_k}{p_k} \cdot p_k d\mu = \int_{\mathbb{R}} q_k d\mu = 1.$$

Furthermore the $X_k$'s are independent with $X_k \geq 0$. Thus with $M_n \equiv \prod_{k=1}^{n} X_k = \prod_{k=1}^{n} \frac{q_k}{p_k}(Y_k)$, the $\{M_n, \mathcal{A}_n\}_{n\geq 1}$ can be viewed as Kakutani's martingale in Example 13.1.14.
(b) In the context given we can let $\mathcal{A}_n \equiv \sigma[Y_1, \ldots, Y_n]$ and take $(\Omega, \mathcal{A})$ of Example 13.1.13 to be $(\mathbb{R}^{\infty}, \mathcal{B}_{\infty})$ Then let $P_n$, $Q_n$ correspond to the densities $\prod_{k=1}^{n} p_k(y_k)$ and $\prod_{k=1}^{n} q_k(y_k)$ on $(\mathbb{R}^n, \mathcal{B}_n)$ for $n \geq 1$. With this notation, $X_n$ of Example 13.1.14 becomes just

$$M_n = \left.\frac{dQ}{dP}\right|_{\mathcal{A}_n} = \prod_{k=1}^{n} \frac{q_k}{p_k}(Y_k),$$

so that $\{M_n, \mathcal{A}_n\}_{n\geq 1}$ is a mean 1 martingale of likelihood ratios.

6. **Bonus problem:** Find the 3rd and 4th order martingales obtained by differentiating the martingale $Y \equiv Y_c$ given in Example 13.1.8 three and four times respectively and setting $c = 0$. (Hint: The Hermite polynomials defined in Exercise 12.7.3, PfS page 325, (11.6.4) page 295, and (11.6.15) page 396, might be useful.)

**Solution:** After some calculation the two martingales turn out to be

$$Z_3(t) = \mathbb{S}(t)^3 - 3t\mathbb{S}(t) \qquad \text{and} \qquad \mathbb{S}(t)^4 - 6t\mathbb{S}^2(t) + 3t^2.$$

Note that $E(Z_3(t)) = 0$ and

$$E(Z_4(t)) = E(\mathbb{S}(t)^4 - 6t^2 + 3t^2 = 6t^2 - 6t^2 = 0.$$