Statistics 522, Problem Set 2 Solutions

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1. Suppose that $\{X_k\}_{n=1}^{\infty}$ are independent and non-negative ($X_k \ge 0$). Show that the following are equivalent:
(i) $\sum_{k=1}^{\infty} X_k < \infty$ almost surely;
(ii) $\sum_{k=1}^{\infty}\{P(X_k > 1) + E(X_k 1_{[X_k \le 1]})\} < \infty$;
(iii) $\sum_{k=1}^{\infty} E(X_k/(1+X_k)) < \infty$.

   Solution: Suppose that (i) holds. By the 3-series theorem it follows that all three series $I_c \equiv \sum_1^{\infty} P(|X_k| > c)$, $II_c \equiv \sum_1^{\infty} Var(X_k^{(c)}) < \infty$; and $III_c \equiv \sum_1^{\infty} E(X_k^{(c)}) < \infty$ where $X_k^{(c)} = X_k 1_{[|X_k| \le c]}$. But since the X_k's are non-negative by taking $c = 1$ this implies that $I_c + III_c = \sum_{k=1}^{\infty}\{P(X_k > 1) + E(X_k 1_{[X_k \le 1]})\} < \infty$, so (ii) holds.
Suppose that (ii) holds. Then

$$\sum_1^{\infty} Var(X_k^{(1)}) \le \sum_{k=1}^{\infty} E(X_k^{(1)})^2 \le \sum_{k=1}^{\infty} E(X_k^{(1)}) = III_{c=1} < \infty,$$

   so all three series converge with $c = 1$. Thus (i) holds by the three series theorem again.
To see that (iii) is equivalent to (ii), let $\psi(x) \equiv x 1_{[x \le 1]} + 1_{[x > 1]}$ and note that

$$\frac{1}{2}\psi(x) \le \frac{x}{1+x} \le \psi(x).$$

   Replacing x by X_k, taking expectations, and then summing on k we get

$$\frac{1}{2}\sum_{k=1}^{\infty} E\psi(X_k) \le \sum_{k=1}^{\infty} E\left(\frac{X_k}{1+X_k}\right) \le \sum_{k=1}^{\infty} E\psi(X_k).$$

   The equivalence of (iii) and (ii) follows from these inequalities.

2. Suppose that $\{Y_k\}_{k=1}^{\infty}$ are independent standard Cauchy (i.e. Cauchy$(0,1)$) random variables.
(a) Does $\sum_{k=1}^{n} 2^{-k} Y_k \to_{a.s.}$ (some rv)S?
(b) For what sequences $\{a_k\}_{k=1}^{\infty}$ does $\sum_{k=1}^{n} a_k Y_k \to_{a.s.}$ (some rv) S?
(c) What is the distribution of the limits S in (a) and (b) (if they

exist)? **Hint:** You may use characteristic functions together with independence here.

Solution: (a) Let $X_k \equiv 2^{-k} Y_k$. We will apply the 3-series theorem. Thus for $c > 0$ we have, since the Cauchy density is $f(y) = \pi^{-1}(1+y^2)^{-2}$ for $y \in \mathbb{R}$,

$$\begin{aligned} P(|X_k| > c) &= P(|Y_k| > 2^k c) = 2 \int_{2^k c}^{\infty} \frac{1}{\pi(1+y^2)} dy \\ &\leq \frac{2}{\pi} \int_{2^k c}^{\infty} y^{-2} dy = \frac{2}{\pi c} 2^{-k}. \end{aligned}$$

Thus

$$\sum_{k=1}^{\infty} P(|X_k| > c) \leq \frac{2}{\pi c} \sum_{k=1}^{\infty} 2^{-k} = \frac{2}{\pi c} < \infty.$$

Now with $X_k^{(c)} \equiv X_k 1_{[|X_k| \leq c]} = 2^{-k} Y_k 1_{[|Y_k| \leq 2^k c]}$ we have

$$E(X_k^{(c)}) = 2^{-k} E\left(Y_k 1_{[|Y_k| \leq 2^k c]}\right) = 2^{-k} \int_{-c2^k}^{c2^k} y f(y) dy = 0$$

by symmetry, and

$$\begin{aligned} Var(X_k^{(c)} &= 2^{-2k} E\left(Y_k^2 1_{[|Y_k| \leq 2^k c]}\right) = 2^{-2k} \int_{-c2^k}^{c2^k} \frac{y^2}{\pi(1+y^2)} dy \\ &\leq \frac{2^{-2k}}{\pi} \cdot 2c2^k = \frac{2c}{\pi} 2^{-k}. \end{aligned}$$

Thus $\sum_{k=1}^{n} E(X_k^{(c)}) = \sum_{k=1}^{n} 0 = 0$ and

$$\sum_{k=1}^{\infty} Var(X_k^{(c)}) \leq \frac{2c}{\pi} \sum_{k=1}^{\infty} 2^{-k} = \frac{2c}{\pi} < \infty.$$

Thus by the three-series theorem it follows that

$$S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} 2^{-k} Y_k \rightarrow_{a.s.} \sum_{k=1}^{\infty} 2^{-k} Y_k \equiv S.$$

(b) If 2^{-k} is replaced by $a_k \geq 0$ with $a_k \to 0$, then arguments paralleling those in (a) yield

$$P(|X_k| > c) \leq \frac{2}{\pi c} a_k, \qquad E(X_k^{(c)}) = 0 \text{ for all } k \geq 1,$$

and

$$Var(X_k^{(c)}) \leq \frac{2}{\pi c} a_k,$$

so the three series all converge if $\sum_{k=1}^{\infty} a_k < \infty$. Thus the three series theorem yields $S_n \equiv \sum_{k=1}^{n} a_k Y_k \rightarrow_{a.s.} S$ for Y_k independent standard Cauchy if and only if $\sum_{k=1}^{\infty} a_k < \infty$.
(c) Note that the results of (b) make sense from the point of view of characteristic functions: since $E \exp(itY_k) = \exp(-|t|)$ for all k and $t \in \mathbb{R}$, since the Y_k's are independent we have

$$\begin{aligned} E(e^{itS_n}) &= \prod_{k=1}^{n} E e^{ita_k Y_k} = \prod_{k=1}^{n} e^{-|t| a_k} \\ &= \exp(-|t| \sum_{k=1}^{n} a_k) \rightarrow \exp(-|t| \sum_{k=1}^{\infty} a_k) \quad \text{if } A \equiv \sum_{k=1}^{\infty} a_k < \infty \\ &= E e^{itAY_1}. \end{aligned}$$

This shows that $S \stackrel{d}{=} AY_1$ where $Y_1 \sim \text{Cauchy}(0,1)$.

3. PfS, Exercise 12.3.1, page 309: Let $Z \sim N(0,1)$, let $\mathbb{V}$, $\mathbb{U}^{(1)}$, and $\mathbb{U}^{(2)}$ be independent Brownian bridge processes, with Z independent of $\mathbb{V}$. Fix $a > 0$. Show that:
(a) $\mathbb{B}(t) = \mathbb{V}(t) + tZ$ is a Brownian motion for $0 \leq t \leq 1$.
(b) $\mathbb{B}(at)/\sqrt{a}$, $0 \leq t < \infty$ is a Brownian motion.
(c) $\mathbb{B}(a+t) - \mathbb{B}(a)$, $t \geq 0$, is a Brownian motion.
(d) $\sqrt{1-a}\,\mathbb{U}^{(1)} \pm \sqrt{a}\,\mathbb{U}^{(2)}$ is a Brownian bridge.
(e) $\mathbb{Z}(t) \equiv \{\mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\}/\sqrt{2}$ is a Brownian bridge, $0 \leq t \leq 1/2$.

Solution: (a) It suffices to show that $\mathbb{B}$ has mean 0 and covariance $s \wedge t$ for $0 \leq s, t \leq 1$. But

$$E\mathbb{B}(t) = E\mathbb{V}(t) + tE(Z) = 0 + t \cdot 0 = 0 \quad \text{for each } t \in [0,1],$$

and, since Z is independent of $\mathbb{V}$

$$\begin{aligned} E(\mathbb{B}(s)\mathbb{B}(t)) &= E\{\mathbb{V}(s) + sZ)(\mathbb{V}(t) + tZ)\} = E\{\mathbb{V}(s)\mathbb{V}(t)\} + stE(Z^2) \\ &= s \wedge t - st + st = s \wedge t. \end{aligned}$$

(b) Note that $E\{\mathbb{B}(at)/\sqrt{a}\} = a^{-1/2}E\mathbb{B}(at) = a^{-1/2}\cdot 0 = 0$ and

$$E\left\{\frac{\mathbb{B}(as)}{\sqrt{a}}\cdot\frac{\mathbb{B}(at)}{\sqrt{a}}\right\} = \frac{1}{a}E\{\mathbb{B}(as)\mathbb{B}(at)\} = \frac{1}{a}\left((as)\wedge(at)\right) = s\wedge t.$$

(c) Let $\mathbb{W}(t) \equiv \mathbb{B}(a+t) - \mathbb{B}(a)$ for $t \geq 0$ and $a > 0$. Then $E\mathbb{W}(t) = E(\mathbb{B}(a+t) - \mathbb{B}(a)) = E\mathbb{B}(a+t) - E\mathbb{B}(a) = 0 - 0 = 0$ and

$$\begin{aligned} E\{\mathbb{W}(s)\mathbb{W}(t)\} &= E\{(\mathbb{B}(a+s)-\mathbb{B}(a))(\mathbb{B}(a+t)-\mathbb{B}(a))\} \\ &= E\{\mathbb{B}(a+s)(\mathbb{B}(a+t)-\mathbb{B}(a))\} \\ &\qquad \text{by independence of } \mathbb{B}(a+t)-\mathbb{B}(a) \text{ and } \mathbb{B}(a) \\ &= E\{\mathbb{B}(a+s)\mathbb{B}(a+t)\} - E\{\mathbb{B}(a+s)\mathbb{B}(a)\} \\ &= (a+s)\wedge(a+t) - (a+s)\wedge a \\ &= a+s-a = s \quad \text{if} \quad s \leq t; \end{aligned}$$

that is $E\{\mathbb{W}(s)\mathbb{W}(t)\} = s\wedge t$.
(d) Let $\mathbb{V}(t) \equiv \sqrt{1-a}\,\mathbb{U}^{(1)} \pm \sqrt{a}\,\mathbb{U}^{(2)}$. Then $E\mathbb{V}(t) = 0$ by linearity of expectation. By independence of $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ we find that

$$\begin{aligned} E\{\mathbb{V}(s)\mathbb{V}(t)\} &= Cov(\sqrt{1-a}\,\mathbb{U}^{(1)}(s) \pm \sqrt{a}\,\mathbb{U}^{(2)}(s), \sqrt{1-a}\,\mathbb{U}^{(1)}(t) \pm \sqrt{a}\,\mathbb{U}^{(2)}(t)) \\ &= (1-a)Cov(\mathbb{U}^{(1)}(s),\mathbb{U}^{(1)}(t)) + aCov(\mathbb{U}^{(2)}(s),\mathbb{U}^{(2)}(t)) \\ &= (1-a)(s\wedge t - st) + a(s\wedge t - st) = s\wedge t - st. \end{aligned}$$

(e) By linearity of expectation, $E\mathbb{Z}(t) = 0$. Furthermore, by independence of $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ we find that for $0 \leq s,t \leq 1/2$

$$\begin{aligned} E\{\mathbb{Z}(s)\mathbb{Z}(t)\} &= Cov(\mathbb{Z}(s),\mathbb{Z}(t)) \\ &= Cov(\mathbb{U}^{(1)}(s) + \mathbb{U}^{(2)}(1-s)\}/\sqrt{2}, \mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\}/\sqrt{2}) \\ &= 2^{-1}\{Cov(\mathbb{U}^{(1)}(s),\mathbb{U}^{(1)}(t)) + Cov(\mathbb{U}^{(2)}(s),\mathbb{U}^{(2)}(t))\} \\ &= 2^{-1}\{s\wedge t - st + (1-s)\wedge(1-t) - (1-s)(1-t)\} \\ &= 2^{-1}\{s - st + (1-t) - ((1-t) - s(1-t)\} \quad \text{if} \quad 0 \leq s \leq t \leq 1/2 \\ &= s(1-t); \end{aligned}$$

Thus $E\{\mathbb{Z}(s)\mathbb{Z}(t)\} = s\wedge t - st$ for $0 \leq s,t \leq 1/2$.

4. (a) In our proof of the existence of Brownian motion as a continuous process on $[0,1]$ we used that fact that the family of Haar functions

$\{g_{nj}: \ 0 \le j \le 2^n - 1, n \ge 0\}$ is a complete orthonormal system for $L_2(0,1)$. Prove the orthonormality part of this assertion. (The completeness will follow easily from martingale theory, so we will address this later.)
(b) In our proof of the existence of Brownian motion as a continuous process on $[0,1]$ we claimed that the integrations and expectations can be interchanged in the computation of the covariance $E\{\mathbb{U}(s)\mathbb{U}(t)\}$. Justify this interchange.

Solution: (a) Recall that $g_{nj}(t) = 2^{n/2} g_{00}(2^n t - j)$ for $j \in \{0, 1, 2, \ldots, 2^n - 1\}$ and $n \ge 1$ where $g_{00}(t) = 2 1_{[0,1/2]}(t) - 1$. Since g_{00} is non-zero only for $t \in [0,1]$, g_{nj} is non-zero only for $j/2^n \le t \le (j+1)/2^n$, and hence $g_{nj}(t) g_{nj'}(t) = 0$ a.e. with respect to Lebesgue measure for any $j' \ne j$. Furthermore, we compute

$$\begin{aligned} \int_0^1 g_{nj}(t) &= \int_0^1 2^{n/2} g_{00}(2^n t - j) dt = 2^{n/2} \int_{j/2^n}^{(j+1)/2^n} g_{00}(2^n t - j) dt \\ &= 2^{n/2} \int_0^{1/2^n} g_{00}(2^n s) ds = 2^{-n/2} \int_0^1 g_{00}(u) du = 2^{n/2}\{1/2 - 1/2\} = 0, \end{aligned}$$

and

$$\begin{aligned} \int_0^1 g_{nj}^2(t) &= \int_0^1 2^n g_{00}^2(2^n t - j) dt = 2^n \int_{j/2^n}^{(j+1)/2^n} g_{00}^2(2^n t - j) dt \\ &= 2^n \int_0^{1/2^n} g_{00}^2(2^n s) ds = \int_0^1 g_{00}^2(u) du = \{1/2 + 1/2\} = 1. \end{aligned}$$

Since $g_{nj}(t) g_{nj'}(t) = 0$ a.e. Lebesgue for $j' \ne j$, we have

$$\int_0^1 g_{nj}(t) g_{nj'}(t) dt = 0 \qquad \text{for} \qquad j' \ne j.$$

Furthermore if $n' \ne n$ and $j' \ne j$, we assume (without loss) that $n' > n$ and $j' > j$. Then the product $g_{n'j'}(t) g_{nj}(t) = 0$ a.e. Lebesgue unless $j/2^n \le j'/2^{n'} < (j'+1)/2^{n'} < (j+1)/2^n$, and then

$$\int_0^1 g_{n'j'}(t) g_{nj}(t) dt = (\pm 1) 2^{n'/2 + n/2} \int_0^1 g_{00}(2^{n'} t - j') 1_{[j'/2^{n'}, (j'+1)/2^{n'}]}(t) dt = 0.$$

Thus the family of Haar functions $\{g_{nj} : \ 0 \le j \le 2^n - 1, n \ge 0\}$ is orthonormal. Is it (together with the constant function 1) complete? To show this we need to show that for any $f \in L_2[0,1]$ we have, with $c_{m,j}(f) \equiv \int_0^1 f(t) g_{m,j}(t) dt$,

$$f_n(t) \equiv \sum_{m=0}^{n} \sum_{j=0}^{2^m - 1} c_{m,j}(f) g_{m,j}(t) \to_2 f(t);$$

i.e. $\int_0^1 (f_n(t) - f(t))^2 dt \to 0$. Several different proofs are possible, but I will postpone the proof for now since it will follow easily via martingale convergence theorems.

(b) Consider $\mathbb{U}(t,\omega) = \sum_{n=0}^{\infty} \mathbb{V}_n(t,\omega)$ where

$$\mathbb{V}_n(t,\omega) = \sum_{j=0}^{2^n - 1} X_{n,j}(\omega) h_{n,j}(t),$$

h_{nj} are the Schauder functions, and X_{nj} are i.i.d. $N(0,1)$. By Fubini's theorem, the claim that $E\mathbb{U}(t) = 0$ will be justified if we show that

$$E\left\{ \sum_{n=0}^{\infty} |\mathbb{V}_n(t)| \right\} < \infty.$$

Now $E|X_{n,j}| = E|Z| = \sqrt{2/\pi}$ for all n and $j \in \{0, \ldots, 2^n - 1\}$. Thus the expectation in the last display is bounded by

$$\begin{aligned} \sum_{n=0}^{\infty} E \sum_{j=0}^{2^n-1} |X_{n,j}||h_{n,j}(t)| &= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} E|X_{n,j}|h_{n,j}(t)| \\ &\le \sum_{n=0}^{\infty} \sqrt{2/\pi} 2^{-n/2} 2^{-1} = (2\pi)^{-1/2} \sum_{n=0}^{\infty} 2^{-n/2} \\ &= (2\pi)^{-1/2} \frac{1}{1 - 2^{-1/2}} = \frac{1}{\sqrt{\pi}(\sqrt{2} - 1)} < \infty. \end{aligned}$$

It follows by Fubini's theorem that

$$E\mathbb{U}(t) = \sum_{n=0}^{\infty} E(\mathbb{V}_n(t)) = \sum_{n=0}^{\infty} 0 = 0.$$

To see that the interchanges can be made in the covariance calculation, note first that by the Cauchy-Schwarz inequality it suffices to show that $E\mathbb{U}^2(t) < \infty$ for each $t \in [0,1]$, since then we have $(E|\mathbb{U}(s)\mathbb{U}(t)|)^2 \le E\{\mathbb{U}^2(s)\}E\{\mathbb{U}^2(t)\} < \infty$. Thus it suffices to show that

$$E\left\{\left(\sum_{n=0}^{\infty}|\mathbb{V}_n(t)|\right)^2\right\} < \infty$$

But by Tonelli's theorem and independence of the $X_{n,j}$'s the left side in the last display equals

$$\begin{aligned}
& E\left\{\sum_{n=0}^{\infty}|\mathbb{V}_n(t)|\cdot\sum_{m=0}^{\infty}|\mathbb{V}_m(t)|\right\} && (1)\\
&= E\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|\mathbb{V}_n(t)||\mathbb{V}_m(t)|\right\} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}E\left\{|\mathbb{V}_n(t)||\mathbb{V}_m(t)|\right\}\\
&= \sum_{n=0}^{\infty}E|\mathbb{V}_n(t)|^2 + \sum_{m\neq n}E\left\{|\mathbb{V}_n(t)||\mathbb{V}_m(t)|\right\}\\
&\le \sum_{n=0}^{\infty}E|\mathbb{V}_n(t)|^2 + \left(\sum_{n=0}^{\infty}E\{|\mathbb{V}_n(t)|\}\right)^2 && (2)
\end{aligned}$$

where, as in our first calculation above,

$$\sum_{n=0}^{\infty}E\{|\mathbb{V}_n(t)|\} \le \frac{1}{\sqrt{\pi}(\sqrt{2}-1)} < \infty$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}E|\mathbb{V}_n(t)|^2 &= \sum_{n=0}^{\infty}E\left(\sum_{j=0}^{2^n-1}X_{nj}h_{nj}(t)\right)\left(\sum_{j'=0}^{2^n-1}X_{nj'}h_{nj'}(t)\right)\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}h_{n,j}^2(t) \le \sum_{n=0}^{\infty}(2^{-n/2-1})^2 = 2^{-1} < \infty.
\end{aligned}$$

It follows that the right side of (2) is finite, and hence $E\mathbb{U}^2(t) < \infty$.

5. PfS, Exercise 3.2.3, page 42 (page 44, 2017): Consider a measure space $(\Omega, \mathcal{A}, \mu)$. Let $\mu_0 \equiv \mu|_{\mathcal{A}_0}$ for a sub $\sigma-$field $\mathcal{A}_0$ of $\mathcal{A}$. Starting with indicator functions, show that $\int X d\mu = \int X d\mu_0$ for any $\mathcal{A}_0-$measurable function X.

Solution: (a) Suppose first that $X = 1_{D^*}$ where $D^* \in \mathcal{A}_0$. Then since $D^* \in \mathcal{A}_0$

$$\int 1_{D^*} d\mu = \mu(D^*) = \mu_0(D^*) = \int 1_{D^*} d\mu_0.$$

Thus the claimed identity holds for indicator functions.
(b) Suppose that $X = \sum_{j=1}^m a_j 1_{D_j}$ for $a_j \in \mathbb{R}$ and $D_j \in \mathcal{A}_0$ for $j = 1, \ldots, m$. Then

$$\begin{aligned}
\int X d\mu &= \int \sum_{j=1}^m a_j 1_{D_j} d\mu = \sum_{j=1}^m a_j \int 1_{D_j} d\mu \\
&= \sum_{j=1}^m a_j \int 1_{D_j} d\mu_0 \quad \text{by part (a)} \\
&= \int \sum_{j=1}^m a_j 1_{D_j} d\mu_0 \quad \text{by linearity of the integral} \\
&= \int X d\mu_0.
\end{aligned}$$

(c) If $X \geq 0$ is $\mathcal{A}_0-$measurable, then there exist simple functions $X_m \nearrow X$ which are also $\mathcal{A}_0-$measurable. Then, by the monotone convergence theorem,

$$\begin{aligned}
\int X d\mu &= \lim_m \int X_m d\mu = \lim_m \int X_m d\mu_0 \quad \text{by part (b)} \\
&= \int X d\mu_0 \quad \text{by monotone convergence again.}
\end{aligned}$$

(d) If X is a general $\mathcal{A}_0$ measurable function, then write $X = X^+ - X^-$ where X^+ and X^- are non-negative. Then by linearity of the integral and (c) it follows that

$$\begin{aligned}
\int X d\mu &= \int (X^+ - X^-) d\mu = \int X^+ d\mu - \int X^- d\mu \\
&= \int X^+ d\mu_0 - \int X^- d\mu_0 \qquad \text{by (c)} \\
&= \int (X^+ - X^-) d\mu_0 = \int X d\mu_0.
\end{aligned}$$