Statistics 522, Problem Set 1 Solutions

Wellner; 1/8/2020

1. Exercise 8.8.1, page 182, PfS. Suppose that $X_1, X_2, \ldots$ are i.i.d. with $P(X_k = 0) = P(X_k = 2) = 1/2$. Show that $\sum_{k=1}^n X_k/3^k \rightarrow_{a.s.}$ (some S), and determine the mean, variance, and the name of the distribution function F_S of S.

   Solution: Note that $X_k \overset{d}{=} 2Y_k$ where the Y_k's are i.i.d. Bernoulli(1/2). Thus $V_k \overset{d}{=} 2Y_k/3^k$ has $\mu_k = E(V_k) = 1/3^k$ and $\sigma_k^2 \equiv Var(V_k) = 1/3^{2k} = 1/9^k$. It follows that $S_n \equiv \sum_{k=1}^n V_k$ has

$$\begin{aligned} E(S_n) &= \sum_{k=1}^n \mu_k \to \sum_1^\infty 1/3^k = \frac{1/3}{1-1/3} = 1/2, \\ Var(S_n) &= \sum_{k=1}^n \sigma_k^2 \to \sum_1^\infty 1/9^k = \frac{1/9}{1-1/9} = 1/8. \end{aligned}$$

   It follows from the 2-series theorem that

$$S_n \equiv \sum_{k=1}^n V_k \rightarrow_{a.s.} S = \sum_{k=1}^\infty V_k = \sum_{k=1}^\infty \frac{1}{3^k} 2Y_k$$

   and that $E(S) = 1/2$ and $Var(S) = 1/8$. In fact, the distribution function F_S of S is the Cantor singular distribution function on $[0,1]$.

2. Exercise 8.8.2, page 182, PfS. (a) Show that $\sum_{k=1}^n a^k X_k \rightarrow_{a.s.}$ (some S) when $X_1, X_2, \ldots$ are independent with $X_k \sim$ Uniform$(-k,k)$ for $k \geq 1$, and where $0 < a < 1$.
   (b) Evaluate the mean and the variance (give a simple expression) of S.

   Solution: First note that if $X_k \sim$ Uniform$(-k,k)$, then $X_k \overset{d}{=} k \cdot 2(U - 1)$ where $U \sim$ Uniform$[0,1]$. Thus we let $Y_k \equiv ka^k 2(U_k - 1)$ where the U_k's are i.i.d. Uniform$(0,1)$, and we find that

$$\begin{aligned} \mu_k &\equiv E(Y_k) = 2ka^k \cdot 0 = 0, \\ \sigma_k^2 &\equiv Var(Y_k) = 4k^2 a^{2k} \cdot \frac{1}{12} = \frac{1}{3} k^2 a^{2k}. \end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=1}^{n} \mu_k &= 0 \to 0,\\ \sum_{k=1}^{n} \sigma_k^2 &= \sum_{k=1}^{n} \frac{1}{3} k^2 a^{2k} \to \sum_{k=1}^{\infty} 3^{-1} k^2 a^{2k} < \infty\end{aligned}$$

since $a \in (0,1)$. Thus by the 2-series theorem $S_n \equiv \sum_{k=1}^{n} Y_k \to_{a.s.} \sum_{k=1}^{\infty} Y_k \equiv S$ where

$$E(S) = 0 \quad \text{and} \quad Var(S) = \frac{1}{3} \sum_{k=1}^{\infty} k^2 a^{2k} = \frac{1}{3} \frac{a^2(1+a^2)}{(1-a^2)^3}$$

via Mathematica (or by easy differentiations of $g(v) \equiv \sum_{k=1}^{\infty} v^k = v/(1-v)$ and some algebra).

3. Exercise 8.8.3, page 182, PfS. Let $X_1, X_2, \ldots$ be arbitrary random variables with all $X_k \geq 0$ a.s. Let $c > 0$ be arbitrary. Then $\sum_{k=1}^{\infty} E(X_k \wedge c) < \infty$ implies that $S_n \equiv \sum_{k=1}^{n} X_k \to_{a.s.}$ (some S). The converse holds for independent random variables.

   Solution: Let $Y_k \equiv Y_k^{(c)} \equiv \min\{X_k, c\} = X_k \wedge c$ for each $k \geq 1$. Note that

   $$c 1_{[X_k > c]} \;\leq\; c 1_{[X_k > c]} + X_k 1_{[X_k \leq c]} = X_k \wedge c$$

   and $P(X_k \neq Y_k) = P(X_k > c)$. Taking expectations on both sides of the inequality in the last display yields

   $$cP(X_k > c) \leq E\{X_k \wedge c\},$$

   and hence it follows that

   $$c \sum_{k=1}^{\infty} P(X_k > c) \leq \sum_{k=1}^{\infty} E\{X_k \wedge c\} < \infty.$$

   This implies that $\sum_1^{\infty} P(X_k \neq Y_k) \leq c^{-1} \sum_1^{\infty} E\{X_k \wedge c\} < \infty$, and hence the sequences $\{X_k\}$ and $\{Y_k\}$ are Khinchine - equivalent. Thus to prove that $S_n \to_{a.s.}$ some S, it suffices (by Proposition 8.2.1(b)) to prove that $T_n \equiv \sum_{k=1}^{n} Y_k \to_{a.s.}$ some T.

For every $1 \leq n < N$, since $Y_k \geq 0$ a.s., it follows that for every $\epsilon > 0$

$$\begin{aligned} P\left(\max_{n\leq m\leq N} |T_m - T_n| \geq \epsilon\right) &\leq P(T_N - T_n \geq \epsilon) \\ &\leq \epsilon^{-1} E(T_N - T_n) \qquad (1) \end{aligned}$$

by Markov's inequality. Here

$$\begin{aligned} T_N - T_n &= \sum_{k=n+1}^{N} Y_k = \sum_{k=n+1}^{N} Y_k \left(1_{[Y_k\leq c]} + 1_{[Y_k>c]}\right) \\ &\leq \sum_{k=n+1}^{N} Y_k 1_{[Y_k\leq c]} + 0 = \sum_{k=n+1}^{N} X_k \wedge c \end{aligned}$$

since $Y_k \equiv X_k \wedge c \leq c$. Taking expectations on both sides of this inequality yields

$$\begin{aligned} E(T_N - T_n) &\leq \sum_{k=n+1}^{N} E(X_k \wedge c) \\ &\leq \sum_{k=n+1}^{\infty} E(X_k \wedge c) \\ &\to 0 \quad \text{as} \quad n \to \infty \qquad (2) \end{aligned}$$

since $\sum_{k=1}^{\infty} E(X_k \wedge c) < \infty$.

Combining (1) and (2) we find that

$$P\left(\max_{n\leq m\leq N} |T_m - T_n| \geq \epsilon\right) \to 0 \quad \text{as} \quad N > n \to \infty$$

and hence $T_n \to_{a.s.}$ some T as $n \to \infty$. Thus $S_n \to_{a.s.}$ some S by Khinchine equivalence.

If the X_k's are independent, let $S_n^{(c)} \equiv \sum_{k=1}^{n} (X_j \wedge c)$. Then $Y_j \equiv X_j \wedge c$ satisfy $|Y_j| \leq c$ a.s., and by (b) of the two-series theorem,

$$\sum_{k=1}^{n} E(Y_k) = \sum_{k=1}^{n} \mu_k \to \mu \equiv \sum_{k=1}^{\infty} E(X_k \wedge c) < \infty,$$

and also

$$\sum_{k=1}^{n} Var(Y_k) = \sum_{k=1}^{n} \sigma_k^2 \to \sum_{k=1}^{\infty} Var(X_k \wedge c) < \infty.$$

4. Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $N(0,1)$ random variables, and let $W_n^2 \equiv \sum_{j=1}^n Z_j^2/(\pi j)^2$. Show that

$$W_n^2 \rightarrow_{a.s.} \sum_{j=1}^\infty \frac{Z_j^2}{(\pi j)^2} \equiv W^2$$

and that $E(W_n^2) \to E(W^2) = 1/6$.

Solution: Let $X_j \equiv Z_j^2/(\pi j)^2$ where the Z_j's are i.i.d. $N(0,1)$. Then $E(Z_j^2) = 1$ and $Var(Z_j^2) = E(Z_j^4) - (E(Z_j^2))^2 = 3 - 1 = 2$, so $\mu_j \equiv E(X_j) = (\pi j)^{-2} E(Z_j^2) = (\pi j)^{-2}$ and $\sigma_j^2 = Var(Z_j^2/(\pi j)^2) = 2/(\pi j)^4$. It follows that

$$\begin{aligned} \mu &= \sum_{j=1}^\infty \mu_j = \pi^{-2} \sum_{j=1}^\infty j^{-2} = \pi^{-2} \cdot \frac{\pi^2}{6} = \frac{1}{6}, \quad \text{and} \\ \sigma^2 &\equiv \sum_{j=1}^\infty \sigma_j^2 = \frac{2}{\pi^4} \sum_{j=1}^\infty j^{-4} = \frac{2}{90} = \frac{1}{45} \end{aligned}$$

since $\sum_{j=1}^\infty j^{-4} = \pi^4/90$. It follows that

$$W_n^2 \equiv S_n = \sum_{j=1}^n X_j \rightarrow_{a.s.} S \overset{d}{=} \sum_{j=1}^\infty \frac{Z_j^2}{\pi^2 j^2}$$

where $E(S) = E(W^2) = 1/6$ and $Var(S) = Var(W^2) = 1/45$. (In fact $W^2 \equiv S \overset{d}{=} \int_0^1 \mathbb{U}^2(t)dt$ so that

$$\begin{aligned} E(W^2) &= E\left(\int_0^1 \mathbb{U}^2(t)dt\right) = \int_0^1 E\{\mathbb{U}^2(t)\}dt \\ &= \int_0^1 t(1-t)dt = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \end{aligned}$$

again.)