

Statistics 522, Final Exam Solutions

Wellner; 3/19/20

1. (40 points). Let X_1, X_2, \dots be independent random variables such that

$$X_n = \begin{cases} n^2 - 1, & \text{with probability } n^{-2}, \\ -1, & \text{with probability } 1 - n^{-2}. \end{cases}$$

Let $S_n \equiv X_1 + \dots + X_n$.

- (i) Prove that $E(X_n) = 0$ for all $n \geq 1$.
- (ii) Show that $n^{-1}S_n \rightarrow_{a.s.} -1$.
- (iii) Show that S_n is a martingale and that S_n^2 is a sub-martingale.
- (iv) Find the predictable variation process $\langle S_n \rangle$ so that $S_n^2 - \langle S_n \rangle$ is a sub-martingale.
- (v) Find a sequence of numbers $\{b_n\}$ such that $b_n^{-1}\langle S_n \rangle \rightarrow_p c$ with $c > 0$.

Solution: (i) First solution:

$$E(X_n) = (n^2 - 1) \cdot n^{-2} + (-1) \cdot (1 - n^{-2}) = 1 - n^{-2} - 1 + n^{-2} = 0.$$

Second solution: Note that $X_n \stackrel{d}{=} (n^2 - 1)Y_n - (1 - Y_n) = n^2Y_n - 1$ where $Y_n \sim \text{Bernoulli}(n^{-2})$. Then $E(X_n) = n^2E(Y_n) - 1 = n^2 \cdot n^{-2} - 1 = 0$.

(ii) Now $P(X_n \neq -1) = n^{-2}$, so $\sum_1^\infty P(X_n \neq -1) = \sum_1^\infty n^{-2} = \pi^2/6 < \infty$. Hence by the first Borel-Cantelli lemma $P(X_n \neq -1 \text{ i.o.}) = 0$. Thus for every $\omega \in A$ with $P(A) = 1$ we have $X_n = -1$ for all $n \geq N_\omega$. Thus for $n \geq N_\omega$ we have

$$\begin{aligned} \frac{S_n}{n} &= \frac{X_1 + \dots + X_{N_\omega}}{n} + \frac{X_{N_\omega+1} + \dots + X_n}{n} \\ &\leq \frac{1^2 - 1 + \dots + N_\omega^2 - 1}{n} + \frac{(n - N_\omega)(-1)}{n} \\ &= \frac{N_\omega(N_\omega + 1)(2N_\omega) + 6N_\omega}{6n} - 1 \end{aligned}$$

where the first term on the right side in the last display can be made arbitrarily small by choosing n sufficiently large. On the other hand $S_n/n \geq -1$ almost surely. Combining these facts yields $n^{-1}S_n \rightarrow_{a.s.} -1$.

(iii) Since the X_n 's are independent with mean 0, it follows easily that $\{S_n, \mathcal{A}_n\}$ is a zero-mean martingale: $E(S_{n+1} | \mathcal{A}_n) = S_n$ a.s. Since $g(x) = x^2$ is convex and the variances of the X_k 's are finite, it also follows that $\{S_n^2, \mathcal{A}_n\}$ is a sub-martingale.

(iv) From general theory we know that for sums of mean zero independent random variables $\{X_k\}$ the predictable variation process of S_n is $\langle S_n \rangle = \sum_{k=1}^n \sigma_k^2$ where $\sigma_k^2 = \text{Var}(X_k)$. Using the binomial representation of the X_k 's given in the solution of (i) we

see that $Var(X_k) = Var(k^2 Y_k) = k^4 \cdot k^{-2}(1 - k^{-2}) = k^2(1 - k^{-2}) = k^2 - 1$, and hence $\{S_n^2 - \langle S \rangle_n, \mathcal{A}_n\}$ is a martingale where

$$\langle S \rangle_n = \sum_{k=1}^n Var(X_k) = \frac{n(n+1)(2n+1)}{6} - n.$$

(v) Let $b_n \equiv n^3$. Then

$$b_n^{-1} \langle S_n \rangle = \frac{n(n+1)(2n+1)}{6n^3} - n^{-2} \rightarrow \frac{2}{6} = \frac{1}{3} > 0.$$

2. (40 points)

Suppose that $Y_1, Y_2, \dots, Y_n, \dots$ are i.i.d. random variables with $E(Y_1) = 0$ and $Var(Y_1) = \sigma^2$. Let $X_{n,i} \equiv a_{n,i} Y_i$ for $1 \leq i \leq n$ where $\{a_{n,i} : 1 \leq i \leq n\}$ are constants. Consider $S_n \equiv \sum_{i=1}^n X_{n,i}$.

(a) Compute $E(S_n)$ and $Var(S_n) \equiv \sigma_n^2$.

(b) Show that if $A_n^2 \equiv \max_{1 \leq i \leq n} |a_{ni}|^2 / \sum_{i=1}^n a_{ni}^2 \rightarrow 0$, then $S_n / \sigma_n \rightarrow_d Z \sim N(0, 1)$.

(c) Now suppose that $a_{n,i} = n^{-1/2}(i/n)^\alpha$ for some $\alpha \in \mathbb{R}$. For what values of α does $S_n \rightarrow_d v_\alpha Z \sim N(0, v_\alpha^2)$ for some $v_\alpha^2 < \infty$? Compute v_α^2 as a function of α .

Solution: (a) We compute $E(X_{ni}) = a_{ni}E(Y_i) = a_{ni} \cdot 0 = 0$ while

$$\sigma_{ni}^2 = Var(X_{ni}) = a_{ni}^2 \sigma^2, \quad \sigma_n^2 = \sum_{i=1}^n \sigma_{ni}^2 = \sigma^2 \sum_{i=1}^n a_{ni}^2.$$

(b) We are in the triangular array framework of the Lindeberg-Feller CLT. It remains only to verify the Lindeberg-Feller condition. We compute

$$\begin{aligned} LF_n(\epsilon) &= \frac{1}{\sigma_n^2} \sum_{i=1}^n E\{X_{ni}^2 1_{\{|X_{ni}| \geq \epsilon \sigma_n\}}\} = \frac{1}{\sigma_n^2} \sum_{i=1}^n a_{ni}^2 E\{Y_i^2 1_{\{|Y_i| \geq \epsilon \sigma_n / |a_{ni}|\}}\} \\ &\leq \frac{1}{\sigma_n^2} \sum_{i=1}^n a_{ni}^2 E\{Y_i^2 1_{\{|Y_i| \geq \epsilon \sigma_n / \max_{1 \leq i \leq n} |a_{ni}|\}}\} = \frac{1}{\sigma_n^2} \sum_{i=1}^n a_{ni}^2 E\{Y_1^2 1_{\{|Y_1| \geq \epsilon \sigma_n / \max_{1 \leq i \leq n} |a_{ni}|\}}\} \\ &\quad \text{since the } Y_i' \text{s are i.i.d.} \\ &= \frac{1}{\sigma^2} E\{Y_1^2 1_{\{|Y_1| \geq \epsilon \sigma_n / \max_{1 \leq i \leq n} |a_{ni}|\}}\} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

by the dominated convergence theorem since $E(Y_1^2) < \infty$ and $A_n = \max_{1 \leq i \leq n} |a_{ni}| / \sqrt{\sum_{i=1}^n a_{ni}^2} \rightarrow 0$. It follows from the Lindeberg - Feller CLT that $S_n / \sigma_n \rightarrow Z \sim N(0, 1)$ with $\sigma_n^2 = \sigma^2 \sum_{i=1}^n a_{ni}^2$.

(c) When $a_{ni} = n^{-1/2}(i/n)^\alpha$ we have

$$\sum_{i=1}^n a_{ni}^2 = n^{-1} \sum_{i=1}^n (i/n)^\alpha \rightarrow \begin{cases} \int_0^1 t^{2\alpha} dt < \infty, & \text{if } \alpha > -1/2, \\ \infty, & \text{if } \alpha \leq -1/2; \end{cases}$$

Note that for $\alpha > -1/2$ we then have

$$A_n^2 = \frac{\max_{1 \leq i \leq n} n^{-1} (i/n)^{2\alpha}}{n^{-1} \sum_{i=1}^n (i/n)^{2\alpha}} \rightarrow 0$$

since the denominator converges to $\int_0^1 t^{2\alpha} dt < \infty$ while the numerator satisfies

$$\begin{aligned} \max_{1 \leq i \leq n} n^{-1} (i/n)^{2\alpha} &= \begin{cases} n^{-1}, & \text{if } \alpha \geq 0, \\ n^{-1-2\alpha} & \end{cases} \\ &\rightarrow 0 \text{ if } \alpha > -1/2. \end{aligned}$$

Thus for $\alpha > -1/2$ we have

$$S_n \rightarrow_d v_\alpha Z \sim N(0, v_\alpha^2)$$

where $v_\alpha^2 = \sigma^2 \int_0^1 t^{2\alpha} dt = \sigma^2 / (2\alpha + 1)$.

3. (40) points Let f be a bounded continuous function on $[0, \infty)$. The Laplace transform of f is the function L on $(0, \infty)$ defined by

$$L(\lambda) \equiv \int_0^{\infty} e^{-\lambda x} f(x) dx. \quad (1)$$

Let X_1, X_2, \dots be independent random variables each with the exponential distribution with rate λ , so $P(X > x) = e^{-\lambda x}$, $E(X) = 1/\lambda$, $Var(X) = 1/\lambda^2$. Let $S_n \equiv X_1 + \dots + X_n$.

- (a) What is the distribution of S_n ?
 (b) Show that

$$E_{\lambda} f(S_n) = (-1)^{n-1} \frac{\lambda^n L^{(n-1)}(\lambda)}{(n-1)!}$$

where $L^{(n-1)}$ denotes the $(n-1)$ st derivative of L .

- (c) Show that $E_{n/y} f(S_n) = E_{n/y} f\left(\frac{nS_n/y}{n/y}\right) = E_1 f\left(\frac{yS_n}{n}\right)$.
 (d) Show that f may be recovered from L as follows: for $y > 0$,

$$f(y) = \lim_{n \uparrow \infty} (-1)^{n-1} \frac{(n/y)^n L^{(n-1)}(n/y)}{(n-1)!}.$$

Hint: For (b) use (a). For (d), use the strong law (or weak law) of large numbers.

Solution: (a) $S_n \sim \text{Gamma}(n, \lambda)$; see PfS, Appendix A, pages 427 - 428.

(b) It follows from (a) that

$$E_{\lambda} f(S_n) = \int_0^{\infty} f(x) \frac{\lambda^n x^{n-1}}{\Gamma(n)} e^{-\lambda x} dx.$$

But with L as defined in (1) we have

$$\begin{aligned} L^{(1)}(\lambda) &= \int_0^{\infty} f(x) \lambda(-x) e^{-\lambda x} dx, \\ L^{(2)}(\lambda) &= \int_0^{\infty} f(x) \lambda(-x)^2 e^{-\lambda x} dx, \\ &\vdots \\ L^{(n-1)}(\lambda) &= \int_0^{\infty} f(x) \lambda(-x)^{(n-1)} e^{-\lambda x} dx, \end{aligned}$$

and hence

$$E_{\lambda} f(S_n) = (-1)^{n-1} \frac{\lambda^n L^{(n-1)}(\lambda)}{(n-1)!}. \quad (2)$$

Note that if $X \sim \exp(\lambda)$, then $\lambda X \sim \exp(1)$. Then, replacing λ by n/y in the last display we see that

$$E_{n/y}f(S_n) = (-1)^{n-1} \frac{(n/y)^n L^{(n-1)}(n/y)}{(n-1)!} \quad (3)$$

where the left side can be written as

$$E_{n/y}f\left(\frac{(n/y)S_n}{n/y}\right) = E_1f\left(\frac{yS_n}{n}\right).$$

But with $\lambda = 1$, $S_n/n \rightarrow 1$ almost surely (or in probability) by the strong (or weak) law of large numbers, and hence the left side of (3) converges to $f(y)$ as $n \rightarrow \infty$.

4. (40 points).

Suppose that Z_1, Z_2, \dots are i.i.d. $N(0, 1)$ rv's. Let $S_n \equiv \sum_{k=1}^n Z_k$, and define

$$Y_n \equiv \exp(aS_n - bn).$$

- (a) For $r \geq 1$, prove that $Y_n \rightarrow_r 0$ if and only if $r < 2b/a^2$.
- (b) When $b = a^2/2$, show that $Y_n = \prod_{j=1}^n X_j$ where the X_j 's are i.i.d. with mean 1.
- (c) Use the theory for Kakutani's martingale to show that when $b = a^2/2$ it follows that $Y_n \rightarrow_{a.s.} 0$.

Solution: (a) Now write $Y_n = \prod_{i=1}^n X_i$ where $X_i \equiv e^{aZ_i - b}$ are i.i.d. with

$$EX_i^r = E \exp(raZ_i) \exp(-rb) = e^{r^2a^2/2 - rb} \equiv \mu_r.$$

Note that $\mu_r < 1$ if and only if $r < 2b/a^2$. Since $Y_n^r = \prod_{i=1}^n X_i^r$, it follows by from the X_i 's being i.i.d. that

$$EY_n^r = \{\mu_r\}^n \rightarrow 0$$

if and only if $r < 2b/a^2$.

(b) From (a) $EX_i = \mu_1 = e^{a^2/2 - b} = e^0 = 1$ when $b = a^2/2$. Thus $Y_n \equiv \prod_{i=1}^n X_i$ is a mean 1 martingale:

$$\begin{aligned} E(Y_{n+1}|X_1, \dots, X_n) &= E(X_{n+1} \prod_{i=1}^n X_i | X_1, \dots, X_n) = \prod_{i=1}^n X_i E(X_{n+1} | X_1, \dots, X_n) \\ &= Y_n \cdot E(X_{n+1}) = Y_n \cdot 1 = Y_n \quad \text{a.s.} \end{aligned}$$

(c) To apply Kakutani's theorem we compute

$$a_n \equiv E(X_n^{1/2}) = \mu_{1/2} = \exp((1/4)a^2/2 - (1/2)a^2/2) = \exp(-a^2/8).$$

Thus $\prod_{n=1}^N a_n = \exp(-Na^2/8) \rightarrow 0$ as $N \rightarrow \infty$; that is, $\prod_{n=1}^{\infty} a_n = 0$. By Kakutani's theorem the martingale Y_n is not closed. Moreover $Y_n \rightarrow_{a.s.} 0$ as $n \rightarrow \infty$.

A second proof of this shows that $Y_n \rightarrow_{a.s.} 0$ if and only if $b > 0$. This proceeds via the SLLN as follows: To prove that $Y_n \rightarrow_{a.s.} 0$ if and only if $b > 0$, first suppose that $b > 0$. Then, by the SLLN,

$$a \frac{S_n}{n} - b \rightarrow_{a.s.} a \cdot 0 - b < 0,$$

so that $aS_n - bn \rightarrow_{a.s.} -\infty$, and hence by the continuous mapping theorem, $Y_n = \exp(aS_n - bn) \rightarrow_{a.s.} 0$.

Now suppose that $Y_n \rightarrow_{a.s.} 0$. Then we have $\log(Y_n) = aS_n - bn \rightarrow_{a.s.} -\infty$; i.e. for any large $M > 0$, there is an $N = N_M(\omega)$ such that for all ω in a set A with probability 1, there is an $N = N_M(\omega)$ so that for $n \geq N$ we have

$$aS_n(\omega) - bn \leq -M \quad \text{for} \quad n \geq N_M(\omega);$$

i.e.

$$a \frac{S_n(\omega)}{n} - b \leq -\frac{M}{n} < 0, \quad n \geq N_M(\omega).$$

Since we know that $S_n/n \rightarrow_{a.s.} 0$, with probability one we can also make $aS_n(\omega)/n$ arbitrarily small for $n > \text{some } N'(\omega)$. Combining these two facts, it is clear that the above inequality can hold only $-b < 0$; i.e. $b > 0$. (Alternatively: since $S_n \sim N(0, n)$, if $b \leq 0$ it follows that

$$P(Y_n \geq 1) = P(aS_n - bn \geq 0) = P(aS_n \geq bn) \geq P(S_n \geq 0) = 1/2,$$

and hence $Y_n \not\rightarrow_{a.s.} 0$.)

5. (40 points)

(a) Suppose that Y is a random variable with values in $[-c, c]$ and with $E(Y) = 0$. Show that for $\theta \in \mathbb{R}$

$$Ee^{\theta Y} \leq \cosh(\theta c) \leq \exp\left(\frac{\theta^2 c^2}{2}\right).$$

(b) Suppose that $\{M_n\}_{n \geq 0}$ is a martingale with $M_0 = 0$ and that for some constants $c_n > 0$ we have $|M_n - M_{n-1}| \leq c_n$ for all $n \geq 1$. Show that

$$P(\sup_{k \leq n} M_k \geq x) \leq \exp\left(-\frac{x^2}{2 \sum_{k=1}^n c_k^2}\right). \quad (4)$$

Hints: (a) Let $f_\theta(z) \equiv \exp(\theta z)$ for $z \in [-c, c]$. Since f is convex

$$f_\theta(y) \leq \frac{c-y}{2c} f_\theta(-c) + \frac{c+y}{2c} f_\theta(c).$$

(b) Recall the proof of an exponential bound in PfS, 8.10.2 (and lecture).

Solution: (a) As noted in the hint, since $f_\theta(y) = e^{\theta y}$ is convex,

$$f_\theta(y) \leq \frac{c-y}{2c} f_\theta(-c) + \frac{c+y}{2c} f_\theta(c).$$

This follows by noting that $y = \lambda c + (1-\lambda)(-c)$ if $\lambda = (c+y)/(2c)$ and $1-\lambda = (c-y)/(2c)$. Thus if Y has $E(Y) = 0$ and $P(|Y| \leq c) = 1$, then

$$\begin{aligned} E f_\theta(Y) &= E e^{\theta Y} \leq E \left(\frac{c+Y}{2c} \right) f_\theta(-c) + E \left(\frac{c-Y}{2c} \right) f_\theta(c) \\ &= \frac{1}{2} f_\theta(c) + \frac{1}{2} f_\theta(-c) \\ &= \cosh(\theta c) \leq e^{\theta^2 c^2 / 2} \end{aligned}$$

where the last inequality follows by comparing the series expansions of the two sides. Alternatively, the last inequality holds if and only if $g(x) \equiv x^2/2 - \log(\cosh(x)) \geq 0$. But this is easily proved by noting that $g(0) = 0$, while $g'(x) = x - \tanh(x)$ and $g''(x) = (\tanh(x))^2 \geq 0$.

(b) Now let $X_k \equiv M_k - M_{k-1}$ for $k = 1, \dots, n$. Then, since $\{e^{\theta M_k}, \mathcal{A}_k\}_{k=1}^n$ is a submartingale and $|X_k| \leq c_k$ almost surely,

$$\begin{aligned} P(\max_{k \leq n} M_k > x) &= P(\max_{1 \leq k \leq n} \sum_{j=1}^k X_j > x) \\ &= P\left(\exp\left(\theta \sum_{j=1}^k X_j\right) > e^{\theta x} \text{ for some } k \leq n\right) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{E \exp\left(\theta \sum_{j=1}^n X_j\right)}{e^{\theta x}} = \frac{EE\{\exp(\theta \sum_{j=1}^n X_j) | \mathcal{A}_{n-1}\}}{e^{\theta x}} \\
&= \frac{E \left\{ \exp\left(\theta \sum_{j=1}^{n-1} X_j\right) E\{\exp(\theta X_n) | \mathcal{A}_{n-1}\} \right\}}{e^{\theta x}} \\
&\leq \frac{E \left\{ \exp(\theta^2 c_n^2/2) \cdot E \exp\{\theta \sum_{j=1}^{n-1} X_j\} \right\}}{e^{\theta x}} \\
&\quad \text{by (a) applied conditionally given } \mathcal{A}_{n-1} \\
&= \dots = \frac{\exp(\theta^2 \sum_{j=1}^n c_j^2/2)}{e^{\theta x}} \equiv \frac{\exp(\theta^2 C_n^2/2)}{e^{\theta x}}
\end{aligned}$$

where $C_n^2 \equiv \sum_{j=1}^n c_j^2$. Minimizing this bound with respect to θ by choosing $\theta = x/C_n^2$ yields the claimed inequality (4). This is Azuma's martingale generalization of Hoeffding's inequality.

6. (40 points).

Let X_1, \dots, X_n be i.i.d. Poisson(1) random variables, set $S_n = X_1 + \dots + X_n$, and $Z_n \equiv (S_n - n)/\sqrt{n}$. Prove Stirling's formula, $n! \sim \sqrt{2\pi n}(n/e)^n$, by showing that each of the following steps is valid. (As usual, $a_n \sim b_n$ for real sequences $\{a_n\}$ and $\{b_n\}$ if and only if $a_n/b_n \rightarrow 1$ as $n \rightarrow \infty$.)

(a) What is the distribution of S_n ?

(b) Show that

$$EZ_n^- = E \left(\frac{S_n - n}{\sqrt{n}} \right)^- = \sum_{k=0}^n \frac{n-k}{\sqrt{n}} \cdot e^{-n} \cdot \frac{n^k}{k!} = \frac{n^{n+1/2} e^{-n}}{n!}.$$

(c) Show that $Z_n \rightarrow_d Z \sim N(0, 1)$.

You may appeal to one of our central limit theorems.

(d) Use (c) and a uniform integrability argument to show that

$$EZ_n^- \rightarrow EZ^- = 1/\sqrt{2\pi}.$$

(e) Now show that $n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n} = \sqrt{2\pi n} (n/e)^n$.

Solution: (a) First, $S_n = X_1 + \dots + X_n \sim \text{Poisson}(n)$, and hence we can compute

$$\begin{aligned}
E \left(\frac{S_n - n}{\sqrt{n}} \right)^- &= \sum_{k=0}^n \left(\frac{n-k}{\sqrt{n}} \right) \frac{n^k e^{-n}}{k!} \\
&= \frac{e^{-n}}{\sqrt{n}} \left\{ \frac{n}{0!} + \sum_{k=1}^n \left(\frac{n^{k+1}}{k!} - \frac{n^k}{(k-1)!} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-n} n^{n+1}}{\sqrt{n} n!} \text{ since the sum telescopes} \\
&= \sqrt{n} e^{-n} \frac{n^n}{n!}.
\end{aligned}$$

(b) Since $E(X_1) = 1$, $Var(X_1) = 1$, the Lindeberg(- Lévy) CLT yields

$$Z_n \equiv \frac{S_n - n}{\sqrt{n}} = \sqrt{n}(\bar{X}_n - 1) \rightarrow_d N(0, 1).$$

(c) Note that $E(Z_n^2) = 1$ for all n , so $\{Z_n^-\}$ is uniformly integrable and it follows that

$$EZ_n^- \rightarrow EZ^- = E(Z^+) = \int_0^\infty z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}}.$$

(d) It follows from (d) and (c) that

$$\frac{n^{1/2}(n/e)^n}{n!} = E(Z_n^-) \rightarrow E(Z^-) = \frac{1}{\sqrt{2\pi}}.$$

This is equivalent to

$$\frac{\sqrt{2\pi n}(n/e)^n}{n!} \rightarrow 1,$$

i.e. $n! \sim \sqrt{2\pi n}(n/e)^n$.