Statistics 522, Problem Set 8

Wellner; 3/04/2020

Reading:

Wellner	Chapter 11, Sections 1-5, pages 1-35.
Shorack, PfS	Sections 9.1-9.3, pages 193-205

Due: Wednesday, March 11, 2020.

Reminder: Final exam, Wednesday, March 18, 2020, 2:30 - 4:20.

1. Exercise 11.6.4, page 34, Wellner, Chapter 11 notes.
   Give a direct proof of the equivalence of (i) and (iv) in Proposition 2.2. (Hint: consider the functions $\psi_\epsilon(x) \equiv \psi(x/\epsilon)$ with ψ as given on page 34.

2. Exercise 9.2.4, PfS Course Notes, page 199. (Exercise 11.8.4, page 293, PfS 2000.)
   Suppose that $\log X \sim N(0,1)$.
   (i) Show that the density of X is given by $f_X(x) = x^{-1}\exp(-(\log x)^2/2)/\sqrt{2\pi}$ for $x > 0$ (and 0 otherwise).
   (ii) For each $a \in [-1,1]$ consider the random variable Y_a with density
   $$f_a(y) = f_X(y)(1 + a\sin(2\pi \log y)) \quad \text{for} \quad y > 0.$$
   Show that $EX^k = EY_a^k$ for all integers $k \geq 1$ and $a \in [-1,1]$.

3. Exercise 11.6.7, page 34, Wellner, Chapter 11 notes. (This asks for a version of the Lindeberg replacement inequality in the case of random vectors X, Y, and W with values in $\mathbb{R}^k$.)

4. Exercise 11.6.9, page 35, Wellner, Chapter 11 notes. Use the Cramér - Wold device to prove the multivariate CLT from the classical (one-dimensional) CLT.

5. **Optional bonus problem 1:** Exercise 11.6.5, page 34, Wellner, Chapter 11 notes.

6. **Optional bonus problem 2:** Give an example of a tight sequence of probability measures $\{P_n\}_{n\geq 1}$ on $(\mathbb{R}, \mathcal{B})$ with exactly three limit points in the sense of convergence in distribution.