Statistics 522, Problem Set 7

Wellner; 2/26/2020

Reading:

Wellner	Chapter 11, Sections 1-2, pages 1-19.
Shorack, PfS	Sections 9.1-9.2, pages 193-200

Due: Wednesday, February 22, 2017.

Reminder: Makeup lecture 3: 2 March (Monday), 9:30 - 10:20, SIG 226

1. Let $\mathbb{S}$ be standard Brownian motion on $[0,\infty)$. For $b > 0$ fixed, let $\tau \equiv \tau_b \equiv \inf\{t > 0 : \mathbb{S}(t) = b\}$. Then τ_b is a stopping time. Use the exponential martingale $Y_r(t) \equiv \exp(r\mathbb{S}(t) - (1/2)r^2 t)$ to give a development for τ_b parallel to that given in class on 13 February for T_b in the context of a simple random walk. That is, use optional sampling of the martingale Y_r to:
   (a) show that $P(\tau_b < \infty) = 1$.
   (b) Show that the Laplace transform of τ_b is given by
   $E\exp(-s\tau_b) = \exp(-\sqrt{2s}b)$ for $s \geq 0$.
   (c) Show that $E(\tau_b) = \infty$.
   (d) The density of τ_b is given by $f_{\tau_b}(t) = (b/t^{3/2})\phi(b/\sqrt{t})1_{(0,\infty)}(t)$ where $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ is the standard normal density. We will show this via reflection arguments for Brownian motion during Spring quarter. Use this expression for the density to show that $E\tau_b^{1/2} = \infty$ and that $E\tau_b^r < \infty$ for each $0 < r < 1/2$.

2. PfS Course notes, Exercise 9.1.2 page 195. (PfS 2000, Exercise 11.7.2, page 289). (Pólya's lemma) If $F_n \to_d F$ for a continuous df F, then
$$\|F_n - F\| = \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \to 0.$$
   Thus if $F_n \to_d F$ with F continuous and $x_n \to x$, then $F_n(x_n) \to F(x)$.

3. PfS Course notes, Exercise 9.1.3 page 195. (PfS 2000, Exercise 11.7.3, page 289). (Verifying tightness) Suppose $X_n \sim F_n$. Show that $\{F_n : n \geq 1\}$ is tight if either: (a) $\limsup_n E|X_n|^r < \infty$ for some $r > 0$; or (b) $F_n \to F$.

4. Exercise 11.6.2, page 34, Wellner, Chapter 11, notes. Suppose that $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$ where both μ and σ^2 are finite. Suppose that $Z \sim P_0$ on $\mathbb{R}$.
   (a) Show that $X_n \stackrel{d}{=} \mu_n + \sigma_n Z \to_d \mu + \sigma Z \stackrel{d}{=} X$.
   (b) Show that for $f \in BL(\mathbb{R})$

$$|Ef(X_n) - Ef(X)| \le \|f\|_{BL} E\{1 \wedge (|\mu_n - \mu| + |\sigma_n - \sigma||Z|)\}.$$

5. **Optional Bonus Problem:** Prove Proposition 2.3, page 9, Wellner, Chapter 11 Notes.