Statistics 522, Problem Set 6

Wellner; 2/19/2020

Reading:
Shorack, PfS Course Notes, Chapter 13, Sections 13.6 - 13.8, pages 374 -387.

Due: Wednesday, February 26, 2020.

Reminder: Makeup lecture 2: 24 February (Monday), 9:30 - 10:20, SIG 226
Makeup lecture 3: 2 March (Monday), 9:30 - 10:20, SIG 226

1. Polyá's urn: At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn, and is replaced together with a new ball of the same color. Just after time n, there are therefore $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n + 1)/(n + 2)$, the proportion of black balls in the urn just after time n. Prove that (relative to a natural filtration which you should specify) M_n is a martingale. Prove that $P(B_n = k) = 1/(n+1)$ for $0 \le k \le n$. What is the distribution of $\Theta \equiv \lim_n M_n$? Prove that for $0 < \theta < 1$,

$$N_n^{\theta} \equiv \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}$$

defines a martingale N_n^{θ}.

2. Exercise 13.3.7, PfS Course Notes, page 359. [Exercise 18.3.7, PfS (2000), page 477.] Let $r > 1$. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a martingale. Then the following are equivalent:
(10) The $|X_n|^r-$process is integrable.
(11) $X_n \to_r X_\infty$
(12) The X_n's are uniformly integrable (thus $X_n \to$ (some X_∞) a.s.) and $X_\infty \in L_r$.
(13) The $|X_n|^r$'s are uniformly integrable.
(14) $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a submg and $E|X_n|^r \nearrow E|X_\infty|^r$.

3. Suppose that $X_1, X_2, \ldots$ are independent random variables on $(\Omega, \mathcal{A})$ and that X_n has density p_n or q_n under P or Q respectively where p_n and q_n are (for simplicity) everywhere positive on $\mathbb{R}$. Let $\mathcal{F} = \sigma[X_1, X_2, \ldots]$ and $\mathcal{F}_n = \sigma[X_1, \ldots, X_n]$ for $n \geq 1$. Let $Y_n \equiv q_n(X_n)/p_n(X_n)$.
(a) Show that

$$M_n \equiv \frac{dQ}{dP}\Big|_{\mathcal{F}_n} = Y_1 \cdots Y_n$$

where the Y_n's are independent and have mean 1 under P; Hence the likelihood ratio martingale of Example 1.14 is the Kakutani product martingale of Example 1.15.
(b) Show that Q is absolutely continuous relative to P on $\mathcal{F}$ if and only if the martingale $\{M_n, \mathcal{F}_n\}$ is uniformly integrable.
(c) Conclude from Kakutani's theorem (PfS Example 4.4, pages 482-483) that $Q \ll P$ on $\mathcal{F}$ if and only if

$$\prod_{n=1}^{\infty} E(Y_n^{1/2}) = \prod_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{p_n(x) q_n(x)}\, dx > 0\,.$$

(d) Construct two examples of sequences p_n and q_n, one in which the condition in (c) holds and one in which it fails. What is the statistical meaning when it holds and when it fails?

4. Suppose that X and Y are non-negative random variables which satisfy the following inequality:

$$\lambda P(X \geq \lambda) \leq E(Y 1_{[X \geq \lambda]}) \quad \text{for every} \quad \lambda > 0. \tag{1}$$

(i) Let $p > 1$ and suppose that $0 < E(X^p) < \infty$ and $E(Y^p) < \infty$. Show that the inequality in the last display implies that

$$\|X\|_p^p = E(X^p) \leq \left(\frac{p}{p-1}\right)^p E(Y^p). \tag{2}$$

(ii) Can you relax the assumption $E(X^p) > 0$ or the assumption $E(Y^p) < \infty$?

5. **Optional bonus problem:** (i) Example 1.11, PfS, course notes, p. 353. Is the claimed covariance formula $s \wedge t - st$ for the process $\mathbb{M}_n$ correct?
   Hint: Note that this has been changed to $s \wedge t$ in PfS (2017), page 348.
   (ii) What would you conjecture as a limit (in distribution) for the processes $n^{-1/2}\mathbb{M}_n(t)$?

6. **Optional bonus problem:** Exercise 13.1.4, page 353, PfS Course Notes, Chapter 13: Verify the claims made in Example 13.1.10, pages 352 and 353. In particular, verify (12), (13), (14), and (15) on pages 352-353.