Statistics 522, Problem Set 5

Wellner; 2/12/2020

Due: Wednesday, February 19, 2020.

Reading: Shorack, PfS (Course Notes),
Chapter 13, sections 18.3 - 18.7, pages 355 - 382

Reminder: Makeup lecture 2: 24 February (Monday), 9:30 - 10:20, SIG 226
Makeup lecture 3: 2 March (Monday), 9:30 - 10:20, SIG 226
Midterm Exam: 14 February (Friday), 11:30 - 12:20 .

1. Suppose that $\{X_n, \mathcal{A}_n\}_{n\ge 0}$ is a martingale, and assume that $\{H_n, \mathcal{A}_n\}_{n\ge 0}$ is a predictable process: i.e. H_n is $\mathcal{A}_{n-1}$ measurable for each n. Then consider the new process $W_n \equiv (H\cdot X)_n \equiv \sum_{k=1}^n H_k(X_k - X_{k-1}) = \sum_{k=1}^n H_k \Delta X_k$.
(i) Show that if $\{H_n\}$ is a bounded (and predictable) process, then $\{W_n, \mathcal{A}_n\}_{n\ge 1}$ is also a martingale.
(ii) Show that the predictable variation process $\langle W\rangle_n$ of W_n is given by
$$\langle W\rangle_n = \langle H\cdot X\rangle_n = \sum_{k=1}^n H_k^2 E\left\{(\Delta X_k)^2 | \mathcal{A}_{k-1}\right\} + H_0^2 E(X_0^2).$$
and hence the sequence
$$L_n \equiv W_n^2 - \langle W\rangle_n \equiv \{(H\cdot X)_n\}^2 - \langle (H\cdot X)\rangle_n$$
is a 0−mean martingale with respect to the $\mathcal{A}_n$'s with $L_0 = H_0^2(X_0^2 - EX_0^2)$.

2. Suppose that $\{X_n, \mathcal{A}_n\}$ and $\{Y_n, \mathcal{A}_n\}$ are sub-martingales. Show that $X_n \vee Y_n = \max\{X_n, Y_n\}$ is a sub-martingale with respect to $\mathcal{A}_n$.

3. Suppose that X and Y are random variables on the probability space $(\Omega, \mathcal{A}, P)$ with $X \in L_2(P)$ and $Y \in L_2(P)$ (so that $XY \in L_1(P)$), and suppose that $\mathcal{D}$ is a sub sigma-field of $\mathcal{A}$. Show that
$$E\{XE(Y|\mathcal{D})\} = E\{E(X|\mathcal{D})Y\} = E\{E(X|\mathcal{D})E(Y|\mathcal{D})\}\,.$$
(With $\langle X, Y\rangle \equiv E(XY)$, this can be rewritten as
$$\langle X, E(Y|\mathcal{D})\rangle = \langle E(X|\mathcal{D}), Y\rangle = \langle E(X|\mathcal{D}), E(Y|\mathcal{D})\rangle\,,$$
and thus is the "self-adjointness" property of the conditional expectation operator.)

4. Exercise 12.4.1, page 313, PfS Course Notes, Chapter 12. (Exercise 12.4.1, page 309, PfS, 2017.)
   Let T_1, $T_2, \ldots$ be (extended) stopping times; no ordering is assumed. Then:
   (1) $T_1 + T_2$ is an extended stopping time if the $\mathcal{A}_t$'s are right-continuous.
   (2) $A \in \mathcal{A}_{T_1}$ implies $A \cap [T_1 \leq T_2] \in \mathcal{A}_{T_2}$. Hint: $[T_1 \wedge t \leq T_2 \wedge t] \in \mathcal{A}_t$.
   $[T_1 < T_2]$, $[T_1 = T_2]$, $[T_1 > T_2]$ are all in both $\mathcal{A}_{T_1}$ and $\mathcal{A}_{T_2}$.
   (3) $T_1 \leq T_2$ implies $\mathcal{A}_{T_1} \subset \mathcal{A}_{T_2}$. Also $\mathcal{A}_{T_1} \cap [T_1 \leq T_2] \subset \mathcal{A}_{T_1 \wedge T_2} = \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$.
   (4) If $T_n \searrow T_0$ and the $\mathcal{A}_t$'s are right continuous, then $\mathcal{A}_{T_0} = \cap_{n=1}^{\infty} \mathcal{A}_{T_n}$.

5. Exercise 13.3.6, PfS Course Notes, page 359. [Exercise 13.3.6, PfS (2017), page 355.]
   Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a sub-martingale with $X_n \geq 0$. Let $r > 1$. Then $\{X_n^r\}$ is uniformly integrable if and only if $\{X_n^r\}$ is integrable.

6. **Bonus problem:** Exercise 13.4.4, PfS Course Notes, page 366. [Exercise 13.4.4, PfS (2017), page 361.]