Statistics 522, Problem Set 4

Wellner; 2/05/2020

Reading:		
Shorack,	PfS Course Notes,	Chapter 8, sections 9-10, pages 186 - 191;
Shorack,	PfS Course Notes,	Chapter 12, section 4, pages 312 - 315;
Shorack,	PfS Course Notes,	Chapter 13, Sections 1-3, pages 349 - 362.
Due:	Wednesday,	February 12, 2020.
Reminder:	Makeup lecture 1:	7 February (Friday), 9:30 - 10:20, SIG 226
	Makeup lecture 2:	24 February (Monday), 9:30 - 10:20, SIG 226
	Makeup lecture 3:	2 March (Monday), 9:30 - 10:20, SIG 226.

1. Let $X_1, X_2, \ldots$ be independent rv's with each $X_k \ge 0$ and $EX_k = 1$. Let $M_n \equiv \prod_{k=1}^n X_k$ for $1 \le k \le n$ with $M_0 \equiv 1$. Then $\{M_n : n \ge 0\}$ is a martingale with all $E(M_n) = 1$.

2. Exercise 13.1.6, page 353, PfS Course Notes, Chapter 13. (Exercise 18.1.6, page 471, PfS, 2000.)
   Find the exponential martingale that corresponds to the martingale Poisson process $\mathbb{M}(t)$ in example 1.12. Then differentiate this twice with respect to c, set $c = 0$ each time, and obtain the two martingales given in the example.

3. Exercise 8.9.2, page 186: In the same context as Example 9.1, turn $\{S_k^2, \mathcal{A}_k\}_{1\le k\le n}$ into a martingale by centering it appropriately.

4. Exercise 8.10.1, page 189. (Exercise 8.11.1, page 249, PfS, 2000.)
   To complete the proof of the Hájek-Rényi inequality for martingales, show that $\{T_k, \mathcal{A}_k\}_{n\le k\le N}$ is a martingale and that $Var(T_N)$ is equal to the quantity in brackets on the right side of (b) on page 188, namely
$$\frac{\sum_{k=1}^n \sigma_k^2}{b_n^2} + \sum_{k=n+1}^N \frac{\sigma_k^2}{b_k^2}.$$

5. Let $Y_1, Y_2, \ldots$ be independent random variables, and suppose that Y_k has either the density p_k or q_k with respect to some common dominating measure μ. Let $X_k \equiv q_k(Y_k)/p_k(Y_k)$ for $k \ge 1$, let P_k denote the

probably measure on $\mathbb{R}^\infty$ corresponding to p_k, and let $P = \prod_{k=1}^{\infty} P_k$ denote the resulting product measure on $(\mathbb{R}^\infty, \mathcal{B}^\infty)$.
(a) Relate the X_k's above to Kakutani's martingale as in Example 13.1.14 (PfS, page 343).
(b) Relate the X_k's above to the likelihood ratio martingale as in Example 13.1.13 (PfS, page 343).

6. **Bonus problem:** Find the 3rd and 4th order martingales obtained by differentiating the martingale $Y \equiv Y_c$ given in Example 13.1.8 three and four times respectively and setting $c = 0$. (Hint: The Hermite polynomials defined in Exercise 12.7.3, PfS page 325, (11.6.4) page 295, and (11.6.15) page 396, might be useful.)