Statistics 522, Problem Set 3
Wellner; 1/22/2020

Reading:

Shorack, PfS	Chapter 8, Section 9, pages 186-187.
Shorack, PfS	Chapter 13, Sections 1-4, pages 349 - 368.

Due: **Wednesday**, February 5, 2020.

Reminder: Jon gone 27, 29 & 31 January (Monday, Wednesday, and Friday)
Makeup lecture 1: 2 February (Friday), 9:30 - 10:20
Makeup lecture 2: 24 February (Monday), 9:30 - 10:20
Makeup lecture 3: 2 March (Monday), 9:30 - 10:20 .

1. Suppose that X and Y are independent random variables with common continuous d.f. F. Let $M \equiv X \vee Y \equiv \max\{X, Y\}$. Show that for each fixed $x \in \mathbb{R}$

$$P(X \le x|M) = 1_{[M \le x]} + \frac{1}{2}\frac{F(x)}{F(M)} 1_{[M > x]}.$$

Hint: Since $\mathcal{M} = \{M \le m : m \in \mathbb{R}]\}$ is a $\overline{\pi}-$ family of sets in $\sigma[M]$, it suffices to show that

$$E\left\{1_{[M \le m]} P(X \le x|M)\right\} = P([M \le m] \cap [X \le x]) = P(M \le m, X \le x)$$

for all $m \in \mathbb{R}$.

2. Suppose that Y is a random variable defined on $(\Omega, \mathcal{A}, P)$ and that $EY^2 < \infty$. Moreover, suppose $\mathcal{D} \subset \mathcal{A}$ is a sub-$\sigma-$field of $\mathcal{A}$, and let $Var(Y|\mathcal{D}) \equiv E\{[Y - E(Y|\mathcal{D})]^2|\mathcal{D}\} = E(Y^2|\mathcal{D}) - [E(Y|\mathcal{D})]^2$.
(a) Show that $Var(Y) = Var\{E(Y|\mathcal{D})\} + E\{Var(Y|\mathcal{D})\}$.
(b) Show that $Z \equiv E(Y|\mathcal{D})$ minimizes $E(Y-Z)^2$ over all $\mathcal{D}-$measurable random variables Z with $E(Z^2) < \infty$.

3. Exercise 7.4.1, page 131, PfS: show that if $\Omega = \sum_i D_i$ for a finite or countable collection of sets D_i, and if $\mathcal{D} \equiv \sigma[D_1, D_2, \ldots]$, then we can take

$$P(A|\mathcal{D}) = \sum_i \frac{P(AD_i)}{P(D_i)} 1_{D_i} \quad (1)$$

where $P(AD_i)/P(D_i) \equiv P(A)$ if $P(D_i) = 0$. Also show that for general $Y \in \mathcal{L}_1$ we can take

$$E(Y|\mathcal{D}) = \sum_i \left\{ \frac{1}{P(D_i)} \int_{D_i} Y dP \right\} 1_{D_i}. \tag{2}$$

4. Exercise 7.4.5, page 139, PfS (2012). If X and Y are independent random variables with mean $\mu_Y = 0$, then for each $r \geq 1$ we have $E|X|^r \leq E|X+Y|^r$. More generally $E|X + \mu_Y|^r \leq E|X+Y|^r$.

5. Exercise 7.4.4, page 139, PfS (2012). (In proving the statement (26), page 136, it is to be understood that $E(XY)$ exists; alternatively, show that the statement holds for all *bounded* $\mathcal{D}-$measurable random variables X.)

6. Exercise 7.4.2, part B, page 134, PfS. Redo the calculations in Example 4.1, page 133, but when the sampling is done without replacement. When the sampling is done without replacement the joint probability distribution for (X_1, X_2) is as follows:

			X_1		
		1	2	3	
	1	0	2/30	3/30	5/30
X_2	2	2/30	2/30	6/30	10/30
	3	3/30	6/30	6/30	15/30
		5/30	10/30	15/30	1

7. Suppose that $X, Y \in L_1(\Omega, \mathcal{F}, P)$ and that $E(Y|X) = X$ a.s. and $E(X|Y) = Y$ a.s. Prove that $P(X = Y) = 1$. (See e.g. Exercise 9.2, Williams, *Probability with Martingales*, page 231.)

8. **Bonus problem:** (Symmetry and conditional expectation). Let $X_1, X_2, \ldots$ be i.i.d. random variables with the same distribution as X where $E|X| < \infty$. Let $S_n \equiv X_1 + \cdots + X_n$, and define

$$\mathcal{G}_n \equiv \sigma\left[S_n, S_{n+1}, \ldots\right] = \sigma\left[S_n, X_{n+1}, X_{n+2}, \ldots\right].$$

Show that $E(X_1|\mathcal{G}_n) = E(X_1|S_n) = n^{-1}S_n$ almost surely. [Hint: Note that $\sigma\left[X_{n+1}, X_{n+2}, \ldots\right]$ is independent of $\sigma\left[X_1, S_n\right]$, and use symmetry to show that $E(1_{[S_n \in B]}X_1) = E(1_{[S_n \in B]}X_2) = \cdots = E(1_{[S_n \in B]}X_n)$.]