Statistics 522, Problem Set 2
Wellner; 1/15/2020

Reading:
Shorack, PfS, Chapter 7, Sections 4-5, (pages 130 - 146, 2012; 134 - 148, 2017)
Shorack, PfS, Chapter 12, Section 3, (pages 308 - 311, 2012; 304 - 307, 2017).

Due: Wednesday, January 22, 2020.

1. Suppose that $\{X_k\}_{n=1}^{\infty}$ are independent and non-negative ($X_k \ge 0$). Show that the following are equivalent:
(i) $\sum_{k=1}^{\infty} X_k < \infty$ almost surely;
(ii) $\sum_{k=1}^{\infty} \{P(X_k > 1) + E(X_k 1_{[X_k \le 1]})\} < \infty$;
(iii) $\sum_{k=1}^{\infty} E(X_k/(1+X_k)) < \infty$.

2. Suppose that $\{Y_k\}_{k=1}^{\infty}$ are independent standard Cauchy (i.e. Cauchy$(0,1)$) random variables.
(a) Does $\sum_{k=1}^{n} 2^{-k} Y_k \rightarrow_{a.s.}$ (some rv)S?
(b) For what sequences $\{a_k\}_{k=1}^{\infty}$ does $\sum_{k=1}^{n} a_k Y_k \rightarrow_{a.s.}$ (some rv) S?
(c) What is the distribution of the limits S in (a) and (b) (if they exist)? **Hint:** You may use characteristic functions together with independence here.

3. PfS, Exercise 12.3.1, page 309: Let $Z \sim N(0,1)$, let $\mathbb{V}$, $\mathbb{U}^{(1)}$, and $\mathbb{U}^{(2)}$ be independent Brownian bridge processes, with Z independent of $\mathbb{V}$. Fix $a > 0$. Show that:
(a) $\mathbb{B}(t) = \mathbb{V}(t) + tZ$ is a Brownian motion for $0 \le t \le 1$.
(b) $\mathbb{B}(at)/\sqrt{a}$, $0 \le t < \infty$ is a Brownian motion.
(c) $\mathbb{B}(a+t) - \mathbb{B}(a)$, $t \ge 0$, is a Brownian motion.
(d) $\sqrt{1-a}\,\mathbb{U}^{(1)} \pm \sqrt{a}\,\mathbb{U}^{(2)}$ is a Brownian bridge.
(e) $\mathbb{Z}(t) \equiv \{\mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\}/\sqrt{2}$ is a Brownian bridge, $0 \le t \le 1/2$.

4. (a) In our proof of the existence of Brownian motion as a continuous process on $[0,1]$ we used that fact that the family of Haar functions $\{g_{nj} : \ 0 \le j \le 2^n - 1, n \ge 0\}$ is a complete orthonormal system for $L_2(0,1)$. Prove the orthonormality part of this assertion. (The completeness will follow easily from martingale theory, so we will address this later.)

(b) In our proof of the existence of Brownian motion as a continuous process on $[0,1]$ we claimed that the integrations and expectations can be interchanged in the computation of the covariance $E\{\mathbb{U}(s)\mathbb{U}(t)\}$. Justify this interchange.

5. PfS, Exercise 3.2.3, page 42 (page 44, 2017): Consider a measure space $(\Omega, \mathcal{A}, \mu)$. Let $\mu_0 \equiv \mu|_{\mathcal{A}_0}$ for a sub $\sigma-$field $\mathcal{A}_0$ of $\mathcal{A}$. Starting with indicator functions, show that $\int X d\mu = \int X d\mu_0$ for any $\mathcal{A}_0-$measurable function X.

6. **Optional bonus problem 1:** (A different representation of a Brownian bridge process) Let $Z_=0, Z_1, \ldots$ be i.i.d. $N(0,1)$ random variables. Let $f_j(t) \equiv \sqrt{2}\sin(j\pi t)$ for $j \geq 1$.
(a) Verify that the functions f_j are orthonormal. Plot the functions f_j for $j \in \{1,2,3\}$.
(b) Show that $\mathbb{U}(t) \equiv \sum_{j=1}^{\infty} Z_j f_j(t)/(j\pi)$, $0 \leq t \leq 1$ is a Brownian bridge process.
(c) Show that $\int_0^1 \mathbb{U}^2(t)dt \stackrel{d}{=} \sum_{j=1}^{\infty} Z_j^2/(j^2\pi^2)$.