Statistics 522, Problem Set 1

Wellner; 1/8/2020

Reading: Shorack, PfS, Chapter 8, Section 8.8, Convergence of Series of Independent RVs (2012-2013), pages 181 - 185; 2017, pages 181 - 184
Shorack, PfS, Chapter 7, Sections 7.4 and 7.5; Conditional Expectation (2012-2013), pages 130 - 146; (2017), pages 134 - 148.

Due: Wednesday, January 15, 2020.

1. Exercise 8.8.1, page 182, PfS. Suppose that $X_1, X_2, \ldots$ are i.i.d. with $P(X_k = 0) = P(X_k = 2) = 1/2$. Show that $\sum_{k=1}^n X_k/3^k \rightarrow_{a.s.}$ (some S), and determine the mean, variance, and the name of the distribution function F_S of S.

2. Exercise 8.8.2, page 182, PfS. (a) Show that $\sum_{k=1}^n a^k X_k \rightarrow_{a.s.}$ (some S) when $X_1, X_2, \ldots$ are independent with $X_k \sim \text{Uniform}(-k, k)$ for $k \geq 1$, and where $0 < a < 1$.
(b) Evaluate the mean and the variance (give a simple expression) of S.

3. Exercise 8.8.3, page 182, PfS. Let $X_1, X_2, \ldots$ be arbitrary random variables with all $X_k \geq 0$ a.s. Let $c > 0$ be arbitrary. Then $\sum_{k=1}^\infty E(X_k \wedge c) < \infty$ implies that $\sum_{k=1}^n X_k \rightarrow_{a.s.}$ (some S). The converse holds for independent random variables.

4. Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $N(0,1)$ random variables, and let $W_n^2 \equiv \sum_{j=1}^n Z_j^2/(\pi j)^2$. Show that

$$W_n^2 \rightarrow_{a.s.} \sum_{j=1}^\infty \frac{Z_j^2}{(\pi j)^2} \equiv W^2$$

and that $E(W_n^2) \rightarrow E(W^2) = 1/6$.

5. **Bonus problem:** Let $Y_1, Y_2, \ldots$ be i.i.d Exponential(1) random variables, and let $0 < \lambda_1 < \cdots < Y_n < \cdots$ where $\sum_{j=1}^{\infty} \lambda_j^{-2} < \infty$ but $\sum_{j=1}^{\infty} \lambda_j^{-1} = \infty$.
(a) Show that

$$S_n \equiv \sum_{j=1}^{n} \frac{(Y_j - 1)}{\lambda_j} \to_{a.s.} \text{(some r.v.) } S < \infty.$$

(b) Find $E(S)$ and $Var(S)$.