Statistics 522, Practice Midterm Exam

Wellner; 2/12/2020

1. (18 points). **Define** *three* of the following four terms:
   (a) The conditional expectation of a random variable X given a (sub-) sigma-field $\mathcal{D}$.
   (b) A martingale, sub-martingale, and super-martingale.
   (c) A stopping time T (relative to a filtration $\mathcal{A}_n$).
   (d) The compensator of a sub-martingale.
   (e) A Brownian motion process $\mathbb{S}$ on $[0, \infty)$.

2. (27 points). Give careful **statements** of *three* of the following five theorems or results:
   (a) The S-mg convergence theorem.
   (b) Doob's decomposition theorem for sub-martinagles.
   (c) The simple optional sampling theorem.
   (d) The step-wise smoothing property of conditional expectations.
   (e) The interpretation of conditional expectations in terms of an (orthogonal) projection onto $L_2(\Omega, \mathcal{G}, P)$ where $\mathcal{G} \subset \mathcal{A}$.

3. (25 points). Suppose that X and Y are random variables on the probability space $(\Omega, \mathcal{A}, P)$ with $X \in L_2(P)$ and $Y \in L_2(P)$ (so that $XY \in L_1(P)$), and suppose that $\mathcal{D}$ is a sub sigma-field of $\mathcal{A}$. Show that

   $$E\{XE(Y|\mathcal{D})\} = E\{E(X|\mathcal{D})Y\} = E\{E(X|\mathcal{D})E(Y|\mathcal{D})\}.$$

   (With $\langle X, Y\rangle \equiv E(XY)$, this can be rewritten as

   $$\langle X, E(Y|\mathcal{D})\rangle = \langle E(X|\mathcal{D}), Y\rangle = \langle E(X|\mathcal{D}), E(Y|\mathcal{D})\rangle,$$

   and thus is the "self-adjointness" property of the conditional expectation operator.)

4. (36 points). Suppose that $X \in L_2(\Omega, \mathcal{A}, P)$ and $\mathcal{D}$ is a sub-sigma field of $\mathcal{A}$. The conditional variance of X given $\mathcal{D}$ is defined by

   $$Var(X|\mathcal{D}) = E\left\{(X - E(X|\mathcal{D}))^2|\mathcal{D}\right\}.$$

   (a) Prove that

   $$Var(X) = E[Var(X|\mathcal{D})] + Var(E(X|\mathcal{D})).$$

   (b) Show that $E(X - Z)^2$ is minimized over all $\mathcal{D}-$measurable random variables Z by $E(X|\mathcal{D})$.
   (c) Interpret the formula in (a) geometrically.

Do one of the following two problems:

5. (42 points). Suppose that $X_0 = 1$, and let $X_n \sim \text{Uniform}(0, X_{n-1})$ for $n \geq 1$. Let $\mathcal{A}_n \equiv \sigma[X_0, X_1, \ldots, X_n]$ for $n = 0, 1, \ldots$.
(a) Show that with $Y_n \equiv 2^n X_n$, $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a martingale, and hence that $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a non-negative super-martingale.
(b) Apply the s-mg convergence theorem to the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$.
(c) There is no convergence theorem stated for a non-negative super-martingale in PfS, but based on what you know about the s-martingale convergence theorem and the reversed martingale convergence theorem, state a convergence theorem for non-negative supermartingales and apply it to $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$. What is the a.s. limit of X_n in the present case?
(d) Is there any connection between the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ and Kakutani's product martingales?
(e) Use (d) to determine whether or not the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is uniformly integrable. Does convergence hold in L_1?
(f) Compute $E(X_{n+1}^2|\mathcal{A}_n)$ and $E(Y_{n+1}^2|\mathcal{A}_n)$.
(g) Use the computation in (f) to find a martingale related to $\{X_n^2\}$, and use it to compute $E(X_n^2)$ and $E(Y_n^2)$. Are either $\{X_n\}$ or $\{Y_n\}$ square-integrable?

6. (42 points) Suppose that $\{Z_n\}_{n=0}^{\infty}$ is a sequence of random variables with

$$P(Z_{n+1} = j|Z_n = i) = e^{-i}\frac{i^j}{j!}, \qquad i, j \in \{0, 1, 2, \ldots\}$$

with the convention that $P(Z_{n+1} = 0|Z_n = 0) = 1$. Also assume that $P(Z_0 = k_0) = 1$ for a fixed (possibly large) integer $k_0 \geq 1$.
(a) Show that $\{Z_n, \mathcal{A}_n\}$ is a martingale with mean k_0 (with respect to the filtration $\{\mathcal{A}_n\}$ with $\mathcal{A}_n = \sigma\{Z_0, \ldots, Z_n\}$ for $n \geq 0$).
(b) Show that with $Y_n \equiv P(Z_{n+1} = 0|\mathcal{A}_n) = P(Z_{n+1} = 0|Z_n)$, the process $\{Y_n, \mathcal{A}_n\}_{n\geq 0}$ is a sub-martingale.
(c) In fact, use Jensen's inequality to show that $\{Y_n, \mathcal{A}_n\}_{n\geq 0}$ is an almost surely strictly increasing sub-martingale.
(d) Use the result of (c) to show that $Y_n \rightarrow_{a.s.} 1$ and hence that $Z_n \rightarrow_{a.s.} 0$.
(e) Find the predictable variation process $\langle Z \rangle_n$ associated with the submartingale $\{Z_n^2, \mathcal{A}_n\}_{n=0}^{\infty}$. Show that $\{Z_n^2 - \langle X \rangle_n\}$ is a zero mean martingale, and use this to compute $E(Z_n^2)$ and $Var(Z_n)$.
(f) Show that for all $\lambda > 0$

$$P(\max_{0\leq n<\infty} Z_n \geq \lambda) \leq \frac{k_0}{\lambda}.$$