Statistics 522, Final Exam

Wellner; 3/11/20

- **This exam is to be completed without any conversation or discussion with any of your classmates.**
- **Please do 5 of the 6 problems stated below.**
- **Please be sure to put your name on your exam.**
- **This exam is due by 12 noon on Monday, 16 March 2020.**
- **Please turn in your exam by any of the following means:**

(i) E-mail a .pdf file to me, at the following address: jonw@uw.edu.
(ii) Put your exam in my mailbox in the Statistics Department, B-313, Padelford Hall.
(iii) Put your completed exam in the cardboard box outside my office door.

1. (40 points). Let $X_1, X_2, \ldots$ be independent random variables such that

$$X_n = \begin{cases} n^2 - 1, & \text{with probability } n^{-2}, \\ -1, & \text{with probability } 1 - n^{-2}. \end{cases}$$

   Let $S_n \equiv X_1 + \cdots + X_n$.
   (i) Prove that $E(X_n) = 0$ for all $n \geq 1$.
   (ii) Show that $n^{-1} S_n \rightarrow_{a.s.} -1$.
   (iii) Show that S_n is a martingale and that S_n^2 is a sub-martingale.
   (iv) Find the predictable variation process $\langle S_n \rangle$ so that $S_n^2 - \langle S_n \rangle$ is a sub-martingale.
   (v) Find a sequence of numbers $\{b_n\}$ such that $b_n^{-1} \langle S_n \rangle \rightarrow_p c$ with $c > 0$.

2. (40 points)
   Suppose that $Y_1, Y_2, \ldots, Y_n, \ldots$ are i.i.d. random variables with $E(Y_1) = 0$ and $Var(Y_1) = \sigma^2$. Let $X_{n,i} \equiv a_{n,i} Y_i$ for $1 \leq i \leq n$ where $\{a_{n,i} : 1 \leq i \leq n\}$ are constants. Consider $S_n \equiv \sum_{i=1}^n X_{n,i}$.
   (a) Compute $E(S_n)$ and $Var(S_n) \equiv \sigma_n^2$.
   (b) Show that if $A_n^2 \equiv \max_{1 \leq i \leq n} |a_{ni}|^2 / \sum_1^n a_{ni}^2 \rightarrow 0$, then $S_n / \sigma_n \rightarrow_d Z \sim N(0, 1)$.
   (c) Now suppose that $a_{n,i} = n^{-1/2} (i/n)^\alpha$ for some $\alpha \in \mathbb{R}$. For what values of α does $S_n \rightarrow_d v_\alpha Z \sim N(0, v_\alpha^2)$ for some $v_\alpha^2 < \infty$? Compute v_α^2 as a function of α.

3. (40) points Let f be a bounded continuous function on $[0, \infty)$. The Laplace transform of f is the function L on $(0, \infty)$ defined by

$$L(\lambda) \equiv \int_0^\infty e^{-\lambda x} f(x) dx.$$

Let $X_1, X_2, \ldots$ be independent random variables each with the exponential distribution with rate λ, so $P(X > x) = e^{-\lambda x}$, $E(X) = 1/\lambda$, $Var(X) = 1/\lambda^2$. Let $S_n \equiv X_1 + \cdots + X_n$.
(a) What is the distribution of S_n?
(b) Show that

$$E_\lambda f(S_n) = (-1)^{n-1} \frac{\lambda^n L^{(n-1)}(\lambda)}{(n-1)!}$$

where $L^{(n-1)}$ denotes the $(n-1)$st derivative of L.
(c) Show that $E_{n/y} f(S_n) = E_{n/y} f\left(\frac{nS_n/y}{n/y}\right) = E_1 f\left(\frac{yS_n}{n}\right)$.
(d) Show that f may be recovered from L as follows: for $y > 0$,

$$f(y) = \lim_{n\uparrow\infty} (-1)^{n-1} \frac{(n/y)^n L^{(n-1)}(n/y)}{(n-1)!}.$$

Hint: For (b) use (a). For (d), use the strong law (or weak law) of large numbers.

4. (40 points).
Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $N(0,1)$ rv's. Let $S_n \equiv \sum_{k=1}^n Z_k$, and define

$$Y_n \equiv \exp(aS_n - bn) .$$

(a) For $r \geq 1$, prove that $Y_n \to_r 0$ if and only if $r < 2b/a^2$.
(b) When $b = a^2/2$, show that $Y_n = \prod_{j=1}^n X_j$ where the X_j's are i.i.d. with mean 1.
(c) Use the theory for Kakutani's martingale to show that when $b = a^2/2$ it follows that $Y_n \to_{a.s.} 0$.

5. (40 points)

(a) Suppose that Y is a random variable with values in $[-c, c]$ and with $E(Y) = 0$. Show that for $\theta \in \mathbb{R}$

$$Ee^{\theta Y} \le \cosh(\theta c) \le \exp\left(\frac{\theta^2 c^2}{2}\right).$$

(b) Suppose that $\{M_n\}_{n\ge 0}$ is a martingale with $M_0 = 0$ and that for some constants $c_n > 0$ we have $|M_n - M_{n-1}| \le c_n$ for all $n \ge 1$. Show that

$$P(\sup_{k\le n} M_k \ge x) \le \exp\left(-\frac{x^2}{2\sum_{k=1}^n c_k^2}\right).$$

Hints: (a) Let $f_\theta(z) \equiv \exp(\theta z)$ for $z \in [-c, c]$. Since f is convex

$$f_\theta(y) \le \frac{c-y}{2c} f_\theta(-c) + \frac{c+y}{2c} f_\theta(c).$$

(b) Recall the proof of an exponential bound in PfS, 8.10.2 (and lecture).

6. (40 points).

Let $X_1, \ldots, X_n$ be i.i.d. Poisson(1) random variables, set $S_n = X_1 + \ldots + X_n$, and $Z_n \equiv (S_n - n)/\sqrt{n}$. Prove Stirling's formula, $n! \sim \sqrt{2\pi n}(n/e)^n$, by showing that each of the following steps is valid. (As usual, $a_n \sim b_n$ for real sequences $\{a_n\}$ and $\{b_n\}$ if and only if $a_n/b_n \to 1$ as $n \to \infty$.)

(a) What is the distribution of S_n?

(b) Show that

$$EZ_n^- = E\left(\frac{S_n - n}{\sqrt{n}}\right)^- = \sum_{k=0}^n \frac{n-k}{\sqrt{n}} \cdot e^{-n} \cdot \frac{n^k}{k!} = \frac{n^{n+1/2}e^{-n}}{n!}.$$

(c) Show that $Z_n \to_d Z \sim N(0, 1)$.
You may appeal to one of our central limit theorems.

(d) Use (c) and a uniform integrability argument to show that

$$EZ_n^- \to EZ^- = 1/\sqrt{2\pi}.$$

(e) Now show that $n! \sim \sqrt{2\pi}n^{n+1/2}e^{-n} = \sqrt{2\pi n}(n/e)^n$.