

Statistics 522, Problem Set 7 Solutions

Wellner; 2/22/2017

1. Let \mathbb{S} be standard Brownian motion on $[0, \infty)$. For $b > 0$ fixed, let $\tau \equiv \tau_b \equiv \inf\{t > 0 : \mathbb{S}(t) = b\}$. Then τ_b is a stopping time. Use the exponential martingale $Y_r(t) \equiv \exp(r\mathbb{S}(t) - (1/2)r^2t)$ to give a development for τ_b parallel to that given in class on 13 February for T_b in the context of a simple random walk. That is, use optional sampling of the martingale Y_r to:
 - (a) show that $P(\tau_b < \infty) = 1$.
 - (b) Show that the Laplace transform of τ_b is given by $E \exp(-s\tau_b) = \exp(-\sqrt{2sb})$ for $s \geq 0$.
 - (c) Show that $E(\tau_b) = \infty$.
 - (d) The density of τ_b is given by $f_{\tau_b}(t) = (b/t^{3/2})\phi(b/\sqrt{t})1_{(0,\infty)}(t)$ where $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ is the standard normal density. We will show this via reflection arguments for Brownian motion during Spring quarter. Use this expression for the density to show that $E\tau_b^{1/2} = \infty$ and that $E\tau_b^r < \infty$ for each $0 < r < 1/2$.

Solution: For any positive integer n , consider the new stopping time $\tau_b \wedge n$. Since Y_c is a mean 1 martingale and $\tau_b \wedge n \leq n$ with $\mathbb{S}(t) \leq b$ for $0 \leq t \leq \tau_b$, it follows by optional sampling that

$$1 = E \exp(r\mathbb{S}(\tau_b \wedge n) - (1/2)r^2(\tau_b \wedge n)).$$

Now $r\mathbb{S}(\tau_b \wedge n) - (1/2)r^2(\tau_b \wedge n) \leq rb$ for $r > 0$, and hence

$$\exp(r\mathbb{S}(\tau_b \wedge n) - (1/2)r^2(\tau_b \wedge n)) \leq \exp(rb).$$

Letting $n \rightarrow \infty$, it follows by the dominated convergence theorem that

$$1 = E \exp(r\mathbb{S}(\tau_b) - (1/2)r^2\tau_b) = E \exp(rb - (1/2)r^2\tau_b). \quad (1)$$

Letting $r \searrow 0$ yields

$$1 = E\{1_{[\tau_b < \infty]}\} = P(\tau_b < \infty).$$

Rearranging (1) yields

$$E \exp(-(1/2)r^2\tau_b) = \exp(-rb). \quad (2)$$

Letting $s \equiv (1/2)r^2$ this becomes

$$L_b(s) \equiv E \exp(-s\tau_b) = \exp(-\sqrt{2sb}). \quad (3)$$

Differentiating this with respect to s gives

$$L'_b(s) = \exp(-\sqrt{2sb}) \cdot (-(2s)^{-1/2}b) \stackrel{s=0}{=} +\infty,$$

which implies that $E\tau_b = \infty$. Furthermore, the Laplace transform in (3) can be inverted to show that τ_b has the claimed density

$$f_{\tau_b}(t) = (b/t^{3/2})\phi(b/\sqrt{t})1_{(0,\infty)}(t).$$

This is more easily derived via reflection arguments for Brownian motion which we will discuss in 523 during Spring quarter. For now we note that for this density

$$\begin{aligned} E\tau_b^{1/2} &= \int_0^\infty t^{1/2} \frac{b}{t^{3/2}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{b^2}{2t}\right) dt \\ &\geq \int_{1/2}^\infty \frac{b}{t} \frac{1}{\sqrt{2\pi}} e^{-b^2} dt = \frac{be^{-b^2}}{\sqrt{2\pi}} \int_{1/2}^\infty t^{-1} dt = \infty. \end{aligned}$$

On the other hand, for $0 < r < 1/2$ we have $3/2 - r > 1$ and hence

$$\begin{aligned} E\tau_b^r &= \int_0^\infty t^r \frac{b}{t^{3/2}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{b^2}{2t}\right) dt \\ &\leq \frac{b}{\sqrt{2\pi}} \left\{ \int_0^1 t^{r-3/2} \exp(-b^2/(2t)) dt + \int_1^\infty t^{-(3/2-r)} dt \right\} \\ &\leq \frac{b}{\sqrt{2\pi}} \left\{ \int_1^\infty \exp(-(b^2/2)s) ds + (3/2 - r - 1)^{-1} \right\} \\ &= \frac{b}{\sqrt{2\pi}} \left\{ \exp(-b^2/2) + (3/2 - r - 1)^{-1} \right\} < \infty. \end{aligned}$$

In fact, via Mathematica,

$$E\tau_b^r = \frac{\Gamma(1/2 - r)}{2^r b^{1-2r} \sqrt{\pi}}.$$

2. PfS Course notes, Exercise 9.1.2 page 195. (PfS 2000, Exercise 11.7.2, page 289).
3. PfS Course notes, Exercise 9.1.2 page 195. (PfS 2000, Exercise 11.7.2, page 289). (Pólya's lemma) If $F_n \rightarrow_d F$ for a continuous df F , then

$$\|F_n - F\| = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \rightarrow 0.$$

Thus if $F_n \rightarrow_d F$ with F continuous and $x_n \rightarrow x$, then $F_n(x_n) \rightarrow F(x)$.

Solution: Let M be a (large) positive integer, and set $x_{M,j} \equiv x_j = F^{-1}(j/(M+1))$ for $j = 1, \dots, M$, and let $x_{M,0} \equiv -\infty$, $x_{M,M+1} \equiv \infty$. Then for $x \in [x_{j-1}, x_j]$ we have

$$F_n(x) - F(x) \begin{cases} \leq F_n(x_j) - F(x_{j-1}) \leq F_n(x_j) - F(x_j) + 1/M \\ \geq F_n(x_{j-1}) - F(x_j) \geq F_n(x_{j-1}) - F(x_{j-1}) - 1/M \end{cases},$$

and hence

$$\begin{aligned} \|F_n - F\| &\leq \max_{1 \leq j \leq M+1} \sup_{x_{j-1} \leq x \leq x_j} |F_n(x) - F(x)| \\ &\leq \max_{1 \leq j \leq M} |F_n(x_j) - F(x_j)| + 1/M \\ &\rightarrow 0 + 1/M \end{aligned}$$

since $F_n(x) \rightarrow F(x)$ at all x in view of F being continuous. But since M is arbitrary, this can be made arbitrarily small; i.e. $\|F_n - F\| \rightarrow 0$.

4. PfS Course notes, Exercise 9.1.3 page 195. (PfS 2000, Exercise 11.7.3, page 289).

Solution: (a) Let $\epsilon > 0$. Now by Markov's inequality

$$\limsup_{n \rightarrow \infty} P(|X_n| > M) \leq \frac{\limsup_{n \rightarrow \infty} E|X_n|^r}{M^r} < \epsilon$$

for $M > M(r, \epsilon) \equiv (\limsup_{n \rightarrow \infty} E|X_n|^r / \epsilon)^{1/r}$. Thus there is an $N \equiv N(\epsilon, r)$ such that $\sup_{n > N} P(|X_n| > M) \leq 2\epsilon$. But since F_1, \dots, F_N are

df's, there exists a $K = K_\epsilon$ so large that $\max_{1 \leq n \leq N} P(|X_n| > K) < 2\epsilon$. Taking $R = R_\epsilon = \max\{M, K\}$ we have

$$\sup_{1 \leq n < \infty} P(|X_n| > R) < 2\epsilon.$$

Thus $\{F_n\}$, the family of distributions of $\{X_n\}$, is tight.

(b) Let $\epsilon > 0$. Let $r = r(\epsilon) \in C_F$ be so large that $1 - F(r) < \epsilon/4$, and let $l = l(\epsilon) \in C_F$ be so small that $F(l) < \epsilon/4$. Now there exists an $N = N_\epsilon$ so large that

$$|F_n(r) - F(r)| < \epsilon/4 \quad \text{for all } n > N$$

and

$$|F_n(l) - F(l)| < \epsilon/4 \quad \text{for all } n > N.$$

Then we have

$$\begin{aligned} \sup_{n > N} F_n([l, r]^c) &\leq F(l) + (1 - F(r)) \\ &\quad + |F_n(l) - F(l)| + |F_n(r) - F(r)| \\ &\leq \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon \end{aligned}$$

But since F_1, \dots, F_N are distribution functions, we can easily find an interval $[l', r']$, such that

$$\max_{1 \leq n \leq N} F_n([l', r']^c) < \epsilon,$$

and hence the interval $[a, b] \equiv [l \wedge l', r \vee r']$ satisfies

$$\sup_{1 \leq n < \infty} F_n([a, b]^c) < \epsilon;$$

i.e. $\{F_n\}$ is tight.

5. Exercise 11.6.2, page 34, Wellner, Chapter 11, notes.

Solution: (b) For $f \in BL(\mathbb{R})$ it follows that $|f(y) - f(x)| \leq \|f\|_{BL} \{1 \wedge |y - x|\}$ for all $x, y \in \mathbb{R}$; recall the inequality before Definition 1.4 on page 4. Thus

$$|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \leq \|f\|_{BL} \{1 \wedge |\mu_n - \mu + (\sigma_n - \sigma)Z|\};$$

This implies that

$$\begin{aligned} |Ef(X_n) - Ef(X)| &= |Ef(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\ &\leq E|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\ &\leq \|f\|_{BL} E\{1 \wedge |\mu_n - \mu| + |\sigma_n - \sigma| |Z|\}. \end{aligned}$$

(a) Since $\mu_n - \mu \rightarrow 0$ and $\sigma_n - \sigma \rightarrow 0$, it follows by the dominated convergence theorem (with dominating function 1) that the right side of the last display converges to 0. Thus $Ef(X_n) \rightarrow Ef(X)$ for all $f \in BL(\mathbb{R})$; therefore $X_n \rightarrow_d X$ by the portmanteau theorem.