Statistics 522, Problem Set 7 Solutions

Wellner; 2/22/2017

1. Let $\mathbb{S}$ be standard Brownian motion on $[0, \infty)$. For $b > 0$ fixed, let $\tau \equiv \tau_b \equiv \inf\{t > 0 : \mathbb{S}(t) = b\}$. Then τ_b is a stopping time. Use the exponential martingale $Y_r(t) \equiv \exp(r\mathbb{S}(t) - (1/2)r^2 t)$ to give a development for τ_b parallel to that given in class on 13 February for T_b in the context of a simple random walk. That is, use optional sampling of the martingale Y_r to:
(a) show that $P(\tau_b < \infty) = 1$.
(b) Show that the Laplace transform of τ_b is given by
$E\exp(-s\tau_b) = \exp(-\sqrt{2s}b)$ for $s \ge 0$.
(c) Show that $E(\tau_b) = \infty$.
(d) The density of τ_b is given by $f_{\tau_b}(t) = (b/t^{3/2})\phi(b/\sqrt{t})1_{(0,\infty)}(t)$ where $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ is the standard normal density. We will show this via reflection arguments for Brownian motion during Spring quarter. Use this expression for the density to show that $E\tau_b^{1/2} = \infty$ and that $E\tau_b^r < \infty$ for each $0 < r < 1/2$.

Solution: For any positive integer n, consider the new stopping time $\tau_b \wedge n$. Since Y_c is a mean 1 martingale and $\tau_b \wedge n \le n$ with $\mathbb{S}(t) \le b$ for $0 \le t \le \tau_b$, it follows by optional sampling that

$$1 = E\exp\left(r\mathbb{S}(\tau_b \wedge n) - (1/2)r^2(\tau_b \wedge n)\right).$$

Now $r\mathbb{S}(\tau_b \wedge n) - (1/2)r^2(\tau_b \wedge n) \le rb$ for $r > 0$, and hence

$$\exp\left(r\mathbb{S}(\tau_b \wedge n) - (1/2)r^2(\tau_b \wedge n)\right) \le \exp(rb).$$

Letting $n \to \infty$, it follows by the dominated convergence theorem that

$$1 = E\exp\left(r\mathbb{S}(\tau_b) - (1/2)r^2\tau_b\right) = E\exp\left(rb - (1/2)r^2\tau_b\right). \tag{1}$$

Letting $r \searrow 0$ yields

$$1 = E\{1_{[\tau_b<\infty]}\} = P(\tau_b < \infty).$$

Rearranging (1) yields

$$E\exp(-(1/2)r^2\tau_b) = \exp(-rb). \tag{2}$$

Letting $s \equiv (1/2)r^2$ this becomes

$$L_b(s) \equiv E\exp(-s\tau_b) = \exp(-\sqrt{2s}b). \tag{3}$$

Differentiating this with respect to s gives

$$L_b'(s) = \exp(-\sqrt{2s}b)\cdot(-(2s)^{-1/2}b) \overset{s=0}{=\!=} +\infty,$$

which implies that $E\tau_b = \infty$. Furthermore, the Laplace transform in (3) can be inverted to show that τ_b has the claimed density

$$f_{\tau_b}(t) = (b/t^{3/2})\phi(b/\sqrt{t})1_{(0,\infty)}(t). \tag{4}$$

This is more easily derived via reflection arguments for Brownian motion which we will discuss in 523 during Spring quarter. For now we note that for this density

$$\begin{aligned} E\tau_b^{1/2} &= \int_0^\infty t^{1/2}\frac{b}{t^{3/2}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{b^2}{2t}\right)dt \\ &\geq \int_{1/2}^\infty \frac{b}{t}\frac{1}{\sqrt{2\pi}}e^{-b^2}dt = \frac{be^{-b^2}}{\sqrt{2\pi}}\int_{1/2}^\infty t^{-1}dt = \infty. \end{aligned}$$

On the other hand, for $0 < r < 1/2$ we have $3/2 - r > 1$ and hence

$$\begin{aligned} E\tau_b^r &= \int_0^\infty t^r\frac{b}{t^{3/2}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{b^2}{2t}\right)dt \\ &\leq \frac{b}{\sqrt{2\pi}}\left\{\int_0^1 t^{r-3/2}\exp(-b^2/(2t))dt + \int_1^\infty t^{-(3/2-r)}dt\right\} \\ &\leq \frac{b}{\sqrt{2\pi}}\left\{\int_1^\infty \exp(-(b^2/2)s)ds + (3/2-r-1)^{-1}\right\} \\ &= \frac{b}{\sqrt{2\pi}}\left\{\exp(-b^2/2) + (3/2-r-1)^{-1}\right\} < \infty. \end{aligned}$$

In fact, via Mathematica,

$$E\tau_b^r = \frac{b^{2r}\Gamma(1/2-r)}{2^r\sqrt{\pi}},$$

and this is confirmed by a direct calculation via the change of variables $v = b^2/(2t)$:

$$\begin{aligned}
E\tau_b^r &= \int_0^\infty t^r \frac{b}{t^{3/2}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{b^2}{2t})dt \\
&= \frac{b}{\sqrt{2\pi}} \int_0^\infty t^{r-3/2} \exp(-\frac{b^2}{2t})dt \\
&= \frac{b}{\sqrt{2\pi}} \int_0^\infty t^{r+1/2} \exp(-\frac{b^2}{2t})t^{-2}dt \\
&= \frac{b}{\sqrt{2\pi}} \int_0^\infty \left(\frac{b^2}{2v}\right)^{r+1/2} \exp(-v)\frac{2}{b^2}dv \\
&= \frac{b}{\sqrt{2\pi}} \left(\frac{b^2}{2}\right)^{r+1/2} \cdot \frac{2}{b^2} \int_0^\infty v^{1/2-r-1}e^{-v}dv \\
&= \frac{b^{2r}}{\sqrt{\pi}2^r}\Gamma(1/2-r).
\end{aligned}$$

Confirmation that the density is given correctly by (4) can be accomplished by computing the Laplace transform: the following solution is due to Hongjian Shi:

$$\begin{aligned}
E\exp(-s\tau_b) &= \int_0^\infty e^{-st} \frac{b}{\sqrt{2\pi}} t^{-3/2} \exp(-\frac{b^2}{2t})dt \\
&= \frac{b}{\sqrt{2\pi}} \int_0^\infty t^{-3/2} \exp\left(-\left(\frac{b^2}{2}\frac{1}{t} + st\right)\right) dt \\
&= \sqrt{\frac{A}{\pi}} \int_0^\infty u^{-3/2} \exp\left(-A\left(u + \frac{1}{u}\right)\right) du \qquad (5) \\
&\qquad \text{by the change of variables} \quad u \equiv (\sqrt{2s}/b)t
\end{aligned}$$

where $A \equiv b\sqrt{s/2}$. But by the change of variables $v = 1/u$ it is seen that

$$\int_0^\infty u^{-3/2} \exp(-A(u + 1/u))du = \int_0^\infty v^{-1/2} \exp(-A(1/v + v))dv,$$

and hence we can write

$$\begin{aligned}
&\int_0^\infty u^{-3/2}\exp(-A(u+1/u))du \\
&= \int_0^\infty \frac{1}{2}\left(\frac{1}{u^{1/2}}+\frac{1}{u^{3/2}}\right)\exp(-A(u+1/u))du \\
&= \int_0^\infty e^{-A(w^2+2)}dw \quad \text{by the change of variables} \quad w \equiv u^{1/2}-u^{-1/2} \\
&= \exp(-2A)\sqrt{\frac{\pi}{A}} \quad \text{by noting that} \\
&\qquad \frac{1}{\sqrt{2\pi 2/A}}\int_0^\infty e^{-Aw^2}dw = 1/2.
\end{aligned}$$

Combining this with (5) yields

$$E\exp(-s\tau) = \exp(-2A) = \exp(-2b\sqrt{s}/\sqrt{2}) = \exp(-b\sqrt{2s}).$$

2. PfS Course notes, Exercise 9.1.2 page 195. (PfS 2000, Exercise 11.7.2, page 289). (Pólya's lemma) If $F_n \to_d F$ for a continuous df F, then

$$\|F_n - F\| = \sup_{x\in\mathbb{R}}|F_n(x) - F(x)| \to 0.$$

Thus if $F_n \to_d F$ with F continuous and $x_n \to x$, then $F_n(x_n) \to F(x)$.

Solution: Let M be a (large) positive integer, and set $x_{M,j} \equiv x_j = F^{-1}(j/(M+1))$ for $j = 1,\ldots,M$, and let $x_{M,0} \equiv -\infty$, $x_{M,M+1} \equiv \infty$. Then for $x \in [x_{j-1}, x_j]$ we have

$$F_n(x) - F(x)\left\{\begin{array}{l} \le F_n(x_j) - F(x_{j-1}) \le F_n(x_j) - F(x_j) + 1/M \\ \ge F_n(x_{j-1}) - F(x_j) \ge F_n(x_{j-1}) - F(x_{j-1}) - 1/M \end{array}\right. ,$$

and hence

$$\begin{aligned}
\|F_n - F\| &\le \max_{1\le j\le M+1}\sup_{x_{j-1}\le x\le x_j}|F_n(x) - F(x)| \\
&\le \max_{1\le j\le M}|F_n(x_j) - F(x_j)| + 1/M \\
&\to 0 + 1/M
\end{aligned}$$

since $F_n(x) \to F(x)$ at all x in view of F being continuous. But since M is arbitrary, this can be made arbitrarily small; i.e. $\|F_n - F\| \to 0$.

The useful corollary is: if $F_n \to_d F$ with F continuous and $x_n \to x$, then $F_n(x_n) \to F(x)$. This follows from Pólya's lemma since

$$\begin{aligned} |F_n(x_n) - F(x)| &\le |F_n(x_n) - F(x_n)| + |F(x_n) - F(x)| \\ &\le \sup_{y \in \mathbb{R}} |F_n(y) - F(y)| + |F(x_n) - F(x)| \\ &\to 0 + 0 = 0 \end{aligned}$$

by continuity of F for the second term and Pólya's lemma for the first term.

3. PfS Course notes, Exercise 9.1.3 page 195. (PfS 2000, Exercise 11.7.3, page 289).

   Solution: (a) Let $\epsilon > 0$. Now by Markov's inequality

   $$\limsup_{n\to\infty} P(|X_n| > M) \le \frac{\limsup_{n\to\infty} E|X_n|^r}{M^r} < \epsilon$$

   for $M > M(r, \epsilon) \equiv (\limsup_{n\to\infty} E|X_n|^r/\epsilon)^{1/r}$. Thus there is an $N \equiv N(\epsilon, r)$ such that $\sup_{n>N} P(|X_n| > M) \le 2\epsilon$. But since $F_1, \ldots, F_N$ are df's, there exists a $K = K_\epsilon$ so large that $\max_{1\le n\le N} P(|X_n| > K) < 2\epsilon$. Taking $R = R_\epsilon = \max\{M, K\}$ we have

   $$\sup_{1\le n<\infty} P(|X_n| > R) < 2\epsilon\,.$$

   Thus $\{F_n\}$, the family of distributions of $\{X_n\}$, is tight.
   (b) Let $\epsilon > 0$. Let $r = r(\epsilon) \in C_F$ be so large that $1 - F(r) < \epsilon/4$, and let $l = l(\epsilon) \in C_F$ be so small that $F(l) < \epsilon/4$. Now there exists an $N = N_\epsilon$ so large that

   $$|F_n(r) - F(r)| < \epsilon/4 \qquad \text{for all} \qquad n > N$$

   and

   $$|F_n(l) - F(l)| < \epsilon/4 \qquad \text{for all} \qquad n > N\,.$$

Then we have

$$\begin{aligned}
\sup_{n>N} F_n([l,r]^c) &\leq F(l) + (1 - F(r)) \\
&\qquad + |F_n(l) - F(l)| + |F_n(r) - F(r)| \\
&\leq \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon
\end{aligned}$$

But since $F_1, \ldots, F_N$ are distribution functions, we can easily find an interval $[l', r']$, such that

$$\max_{1 \leq n \leq N} F_n([l', r']^c) < \epsilon \,,$$

and hence the interval $[a, b] \equiv [l \wedge l', r \vee r']$ satisfies

$$\sup_{1 \leq n < \infty} F_n([a, b]^c) < \epsilon \,;$$

i.e $\{F_n\}$ is tight.

4. Exercise 11.6.2, page 34, Wellner, Chapter 11, notes.

Solution: (b) For $f \in BL(\mathbb{R})$ it follows that $|f(y) - f(x)| \leq \|f\|_{BL}\{1 \wedge |y - x|\}$ for all $x, y \in \mathbb{R}$; recall the inequality before Definition 1.4 on page 4. Thus

$$|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \leq \|f\|_{BL}\{1 \wedge |\mu_n - \mu + (\sigma_n - \sigma) Z|\};$$

This implies that

$$\begin{aligned}
|Ef(X_n) - Ef(X)| &= |Ef(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\
&\leq E|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\
&\leq \|f\|_{BL} E\{1 \wedge |\mu_n - \mu| + |\sigma_n - \sigma||Z|\}.
\end{aligned}$$

(a) Since $\mu_n - \mu \to 0$ and $\sigma_n - \sigma \to 0$, it follows by the dominated convergence theorem (with dominating function 1) that the right side of the last display converges to 0. Thus $Ef(X_n) \to Ef(X)$ for all $f \in BL(\mathbb{R})$; therefore $X_n \to_d X$ by the portmanteau theorem.