Statistics 522, Problem Set 6 Solutions

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1. Polyá's urn: At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn, and is replaced together with a new ball of the same color. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n+1)/(n+2)$, the proportion of black balls in the urn just after time n. Prove that (relative to a natural filtration which you should specify) M_n is a martingale. Prove that $P(B_n = k) = 1/(n+1)$ for $0 \le k \le n$. What is the distribution of $\Theta \equiv \lim_n M_n$? Prove that for $0 < \theta < 1$,

$$N_n^\theta \equiv \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}$$

defines a martingale N_n^θ.

Solution: Let $\mathcal{F}_n \equiv \sigma(B_1, \ldots, B_n)$. Note that $M_n \equiv (B_n+1)/(n+2)$ is the conditional (given $\mathcal{F}_n$) probability of drawing a black ball at the $n+1$st draw. Thus we compute

$$\begin{aligned} E(M_{n+1}|\mathcal{F}_n) &= E\left(\frac{B_{n+1}+1}{n+3}|\mathcal{F}_n\right) = \frac{1}{n+3}E(B_{n+1}+1|\mathcal{F}_n) \\ &= \frac{1}{n+3}\{(B_n+1)(1-M_n) + (B_n+2)M_n\} \\ &= \frac{1}{n+3}\{B_n + 1 - M_n + 2M_n\} \\ &= \frac{1}{n+3}\{(n+2)M_n + M_n\} = M_n \qquad \text{a.s.}. \end{aligned}$$

Hence $\{M_n, \mathcal{F}_n\}$ is a martingale. Similarly, letting

$$p_n(k) \equiv \frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k},$$

the process $N_n^\theta = p_n(B_n)$ and

$$\begin{aligned}
E(N_{n+1}^\theta|\mathcal{F}_n) &= E(p_{n+1}(B_{n+1})|\mathcal{F}_n) \\
&= p_{n+1}(B_n)(1-M_n) + p_{n+1}(B_n+1)M_n \\
&= \frac{(n+2)!}{B_n!(n+1-B_n)!}\theta^{B_n}(1-\theta)^{n+1-B_n}\frac{(n+1-B_n)}{(n+2)} \\
&\qquad + \frac{(n+2)!}{(B_n+1)!(n+1-B_n-1)!}\theta^{B_n+1}(1-\theta)^{n+1-B_n-1}\frac{(B_n+1)}{(n+2)} \\
&= \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\{(1-\theta)+\theta\} \\
&= p_n(B_n) \equiv N_n^\theta \qquad \text{a.s.},
\end{aligned}$$

so $\{N_n^\theta, \mathcal{F}_n\}$ is a martingale. This implies that $EN_n^\theta = EN_0^\theta = 1$ for all $\theta \in (0,1)$, or

$$E\left\{\frac{n!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\right\} = \frac{1}{n+1}\,. \tag{1}$$

This equality clearly holds if $P(B_n = k) = 1/(n+1)$ for $k = 0, \ldots, n$. On the other hand, (1.1) implies, by letting $\alpha = \theta/(1-\theta)$, that, with $p_k = P(B_n = k)$,

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\alpha^k p_k = \frac{1}{n+1}(1+\alpha)^n = \frac{1}{n+1}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\alpha^k\,,$$

and this yields $p_k = 1/(n+1)$ by matching coefficients.

The distribution of B_n is a discrete uniform distribution on $0, \ldots, n$ for every n, so the distribution of M_n is a discrete uniform distribution on $0 < 1/(n+2) < \cdots < (n+1)/(n+2) < 1$ and it is clear that $M_n \to_d U(0,1)$ as $n \to \infty$; $P(M_n \le u) = [(n+2)u]/(n+1) \to u = P(U \le u)$ where $U \sim \text{Uniform}(0,1)$.

2. Suppose that $X_1, X_2, \ldots$ are independent random variables on $(\Omega, \mathcal{A})$ and that X_n has density p_n or q_n under P or Q respectively where p_n and q_n are (for simplicity) everywhere positive on $\mathbb{R}$. Let $\mathcal{F} = \sigma[X_1, X_2, \ldots]$ and $\mathcal{F}_n = \sigma[X_1, \ldots, X_n]$ for $n \ge 1$. Let $Y_n \equiv q_n(X_n)/p_n(X_n)$.
(a) Show that

$$M_n \equiv \frac{dQ}{dP}\Big|_{\mathcal{F}_n} = Y_1 \cdots Y_n$$

where the Y_n's are independent and have mean 1 under P; Hence the likelihood ratio martingale of Example 1.14 is the Kakutani product martingale of Example 1.15.
(b) Show that Q is absolutely continuous relative to P on $\mathcal{F}$ if and only if the martingale $\{M_n, \mathcal{F}_n\}$ is uniformly integrable.
(c) Conclude from Kakutani's theorem (PfS Example 4.4, pages 482-483) that $Q \ll P$ on $\mathcal{F}$ if and only if

$$\prod_{n=1}^{\infty} E(Y_n^{1/2}) = \prod_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{p_n(x) q_n(x)}dx > 0 \,.$$

(d) Construct two examples of sequences p_n and q_n, one in which the condition in (c) holds and one in which it fails. What is the statistical meaning when it holds and when it fails?

Solution: (a) Let $A_i \in \sigma(X_i)$ for $i = 1, \ldots, n$. Then

$$\begin{aligned}
E_P\{1_{A_1\times\cdots\times A_n}\frac{dQ}{dP}\} &= E_Q\{1_{A_1\times\cdots\times A_n}\} \\
&\qquad \text{by definition of the Radon-Nikodym derivative} \\
&= \prod_{i=1}^{n} E_Q(1_{A_i}) \qquad \text{by independence} \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} 1_{A_i}(x) q_i(x) d\mu(x) \qquad \text{by existence of the densities } q_i \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} 1_{A_i}(x) \frac{q_i(x)}{p_i(x)} p_i(x) d\mu(x) \\
&= \prod_{i=1}^{n} E_P\{1_{A_i} Y_i\} \\
&= E_P\{1_{A_1\times\cdots\times A_k} Y_1 \cdots Y_n\} \qquad \text{by independence}\,.
\end{aligned}$$

Now $Y_1 \cdots Y_n$ is $\mathcal{F}_n$ measurable (since it is a function of $X_1, \ldots, X_n$ and agrees with $dQ/dP|_{\mathcal{F}_n}$ on the $\overline{\pi}-$ system $\sigma(X_1) \times \cdots \times \sigma(X_n)$. Thus the claimed equality holds. The Y_i's are independent because the X_i's are independent and they have mean 1 because

$$E_P Y_i = \int_{\mathbb{R}} \frac{q_i(x)}{p_i(x)} p_i(x) d\mu(x) = \int_{\mathbb{R}} q_i(x) dx = 1\,.$$

(b) If $Q \ll P$, with Radon-Nikodym derivative $dQ/dP \equiv Z$, then $M_n = E(Z|\mathcal{F}_n)$ with $E_P(Z) = Q(\mathbb{R}^\infty) = 1$, so $\{M_n, \mathcal{F}_n\}_{n=0}^{\overline{\infty}}$ is a martingale closed at infinity and is uniformly integrable. Conversely, if $\{M_n\}$ is uniformly integrable, then $M_n \to_{a.s.} M_\infty$ and $E(M_\infty|\mathcal{F}_n) = M_n$ almost surely for every n. Now consider the measures Q and $\tilde{Q}$ defined by

$$\tilde{Q}(A) = E\{1_A M_\infty\}\,.$$

These measures agree on the π-system $\cup\mathcal{F}_n$, and hence they agree on $\mathcal{F} = \sigma(\cup\mathcal{F}_n)$. This implies (by the pi-lambda theorem) that Q and $\tilde{Q}$ agree on $\mathcal{F}$, and hence $M_\infty = dQ/dP$ on $\mathcal{F}$, and $Q \ll P$.
(c) By Kakutani's theorem we conclude that Q is absolutely continuous with respect to P on $\mathcal{F}$ if and only if

$$\prod_{n=1}^\infty E(Y_n^{1/2}) = \prod_{n=1}^\infty \int_{\mathbb{R}} (q_n(x)/p_n(x))^{1/2} p_n(x)dx = \prod_{n=1}^\infty \int_{\mathbb{R}} \sqrt{p_n(x)q_n(x)}dx > 0\,.$$

Since this is symmetric in p_n and q_n and since these densities are everywhere positive, we also can conclude that P is absolutely continuous with respect to Q on $\mathcal{F}$; thus Q and P are mutually absolutely continuous or *equivalent* on $\mathcal{F}$.
(d) Suppose that $p_n(x) = \exp(-x)1_{[0,\infty)}(x)$ and $q_n(x) = \lambda_n \exp(-\lambda_n x)1_{[0,\infty)}(x)$ with $\lambda_n = 1 + c_n$ where $c_n \to 0$. Then we compute

$$E(Y_n^{1/2}) = \int_0^\infty \lambda_n^{1/2} \exp(-(1+\lambda_n)x/2)dx = \frac{2\lambda_n^{1/2}}{1+\lambda_n}\,,$$

and

$$\begin{aligned} H^2(P_n, Q_n) &= \frac{1}{2}\int (\sqrt{p_n(x)} - \sqrt{q_n(x)})^2 dx = 1 - E(Y_n^{1/2}) \\ &= 1 - \frac{2\lambda_n^{1/2}}{1+\lambda_n} \\ &= \frac{1+\lambda_n - 2\lambda_n^{1/2}}{1+\lambda_n} \\ &= \frac{2 + c_n - 2(1+c_n)^{1/2}}{2+c_n} \\ &\sim \frac{1}{8}c_n^2 \qquad \text{as} \quad n \to \infty \end{aligned}$$

since $c_n \to 0$ and $(1+c_n)^{1/2} = 1 + (1/2)c_n - (1/8 + o(1))c_n^2$. Thus if $c_n = n^{-r}$ with $r > 1/2$ it follows that

$$\sum_1^\infty (1 - E(Y_n^{1/2})) = \sum_1^\infty H^2(P_n, Q_n) < \infty\,,$$

and $Q \ll P$ on $\mathcal{F}$. If $c_n = n^{-1/2}$, then

$$\sum_1^\infty (1 - E(Y_n^{1/2})) = \sum_1^\infty H^2(P_n, Q_n) = \infty\,,$$

and by Kakutuni's theorem we conclude that $M_\infty = 0$ almost surely P. In this case Q and P are singular on $\mathbb{R}^\infty$: there is a set $A \subset \mathbb{R}^\infty$ such that $Q(A) = 1$ and $P(A) = 0$; i.e. $P(A^c) = 1$.

3. Let $X_1, X_2, \ldots$ be i.i.d. rv's with $P(X = 1) = p$, $P(X = -1) = 1 - p \equiv q$, where $0 < p < 1$ and $p \neq q$. Suppose that a, b are integers with $-a < 0 < b$. Define

$$S_n = X_1 + \ldots + X_n, \qquad T \equiv \inf\{n : S_n = -a, \text{or} \;\; S_n = b\}\,.$$

Let $\mathcal{F}_n \equiv \sigma[X_1, \ldots, X_n]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prove that $M_n \equiv (q/p)^{S_n}$ and $N_n = S_n - n(p-q)$ define martingales M_n and N_n. How would you use these martingales to deduce the values of $P(S_T = -a)$ and $E(S_T)$? [Hint: see PfS, Course Notes, pages 381-382.]

Solution: T is clearly a stopping time and, for each n and $T \wedge n$ is a bounded stopping time for each n. Since M_n is a mean 1 martingale, we conclude by optional sampling that

$$1 = EM_0 = EM_{T\wedge n} = E\{(q/p)^{S_{T\wedge n}}\}. \tag{2}$$

Now consider the case $0 < p < 1/2$ so that $q > p$ and $\mu \equiv p - q < 0$. Now $-a \le S_{T\wedge n} \le b$ with $b \ge 1$ and $-a \le -1$ so $(q/p)^{S_{T\wedge n}} \le (q/b)^b$ for all n. Thus by the dominated convergence theorem we conclude from (2) that

$$1 = EM_0 = E\{(q/p)^{S_T}\}.$$

In the case $1/2 < p$, we have $p > q$ and $\mu = p - q > 0$, and again $-a \le S_{T\wedge n} \le b$, but now $M_{T\wedge n} = (q/p)^{S_{T\wedge n}} = (p/q)^{-S_{T\wedge n}} \le (p/q)^a$, and we again conclude from (2) and the dominated convergence theorem that (3) holds.

Alternatively, we can proceed by verifying the hypotheses of Williams PwM, E10.5, page 233. To see this, note that

$$P(T \le n + b|\mathcal{F}_n) \ge p^{b-S_n} + q^{S_n} \ge (p \wedge q)^b \equiv \epsilon > 0$$

since $p \in (0,1)$. Thus the hypotheses of Williams E10.5 hold with $N = b$ and $\epsilon \equiv (p \wedge q)^b$. Thus $E(T) < \infty$, and the third set of sufficient conditons for Doob's optional sampling theorem hold. Since $\{S_n - n(p-q), \mathcal{F}_n\}$ and $\{(q/p)^{S_n}, \mathcal{F}_n\}$ are both martingales, we again conclude from Doob's optional sampling theorem that (3) holds.

Now the right side of (3) equals

$$\left(\frac{q}{p}\right)^b P(S_T = b) + \left(\frac{q}{p}\right)^{-a} P(S_T = -a) \equiv \left(\frac{q}{p}\right)^b p_b + \left(\frac{q}{p}\right)^{-a} (1 - p_b) \, .$$

Thus we can solve for p_b to obtain

$$p_b = P(S_T = b) = \frac{1 - (q/p)^{-a}}{(q/p)^b - (q/p)^{-a}} = \frac{(q/p)^a - 1}{(q/p)^{a+b} - 1} \, ,$$

and

$$p_a = P(S_T = -a) = 1 - p_b = \frac{(q/p)^b - 1}{(q/p)^b - (q/p)^{-a}} = \frac{1 - (p/q)^b}{1 - (p/q)^{a+b}} \, .$$

It follows that

$$\begin{aligned} E(S_T) &= bp_b + (-a)p_a \\ &= b\left(1 - \frac{1 - (p/q)^b}{1 - (p/q)^{a+b}}\right) - a\frac{1 - (p/q)^b}{1 - (p/q)^{a+b}} \\ &= b - (a+b)\frac{1 - (p/q)^b}{1 - (p/q)^{a+b}} \, . \end{aligned}$$

Since $\{S_n - n(p-q)\} = \{S_n - n\mu\}$ is a martingale, we deduce that $E(S_T - T\mu) = 0$ and hence that

$$E(T) = \frac{1}{\mu}E(S_T) = \frac{1}{\mu}\left\{b - (a+b)\frac{1 - (p/q)^b}{1 - (p/q)^{a+b}}\right\} \, .$$

You should also take a look at the situation for $p = q = 1/2$ in Section 18.7, page 499.

4. Exercise 13.7.2, PfS, Course Notes page 382. Suppose that S_μ is Brownian motion with drift: $S_\mu(t) = S(t) + \mu t$ for $t \geq 0$. Let $\tau_{ab} \equiv \tau \equiv \inf\{t \geq 0: \ S_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$.
Claim 1: $S_0(t)$, $S_0^2(t) - t$, $S_\mu(t) - \mu t$ are mean 0 martingales, and, with $\theta = -2\mu$,

$$\exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$$

is a mean 1 martingale.
Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a+b)$ and $E\tau = ab$.
Claim 3: If $\mu \neq 0$, then

$$P(S(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$

and

$$E(\tau) = \frac{b}{\mu} - \frac{a+b}{\mu}\frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}} .$$

Claim 4: If $\mu < 0$, then with $\|S_\mu^+\|_0^\infty \equiv \sup_{0 \leq t < \infty} S_\mu(t)$, $P(\|S_\mu^+\|_0^\infty \geq b) = \exp(-2|\mu|b)$ for all $b > 0$; i.e. $\|S_\mu^+\|_0^\infty \sim$ Exponential$(2|\mu|)$.
(Note the analogies with problem 2.)

Solution: Suppose that S_μ is Brownian motion with drift: $S_\mu(t) = S(t) + \mu t$ for $t \geq 0$. Let $\tau_{ab} \equiv \tau \equiv \inf\{t \geq 0: \ S_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$.
Claim 1: $S_0(t)$, $S_0^2(t) - t$, $S_\mu(t) - \mu t$ are mean 0 martingales, and, with $\theta = -2\mu$,

$$\exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$$

is a mean 1 martingale.
Proof of claim 1: Since standard Brownian motion S has independent increments, with $\mathcal{A}_t \equiv \sigma[S(s), 0 \leq s \leq t]$ we have, for $0 \leq s \leq t$,

$$\begin{aligned} E(S(t)|\mathcal{A}_s) &= E(S(t) - S(s) + S(s)|\mathcal{A}_s) \\ &= E(S(t) - S(s)|\mathcal{A}_s) + E(S(s)|\mathcal{A}_s) \\ &= E(S(t) - S(s)) + S(s) = 0 + S(s) = S(s) \qquad \text{a.s.} \end{aligned}$$

so that $\{S(t), \mathcal{A}_t\}_{t\geq 0}^\infty$ is a zero - mean martingale. Since $S_\mu(t) - \mu t = S_0(t) = S(t)$, it follows immediately that $\{S_\mu(t) - \mu t, \mathcal{A}_t\}_{t\geq 0}^\infty$ is also a 0-mean martingale. To see that $\{S^2(t) - t, \mathcal{A}_t\}_{t\geq 0}^{\infty}$ is a zero-mean martingale, we calculate

$$\begin{aligned}
E(S^2(t) - t|\mathcal{A}_s) &= E([S(t) - S(s) + S(s)]^2 - (t - s + s)|\mathcal{A}_s) \\
&= E((S(t) - S(s))^2 - (t - s)|\mathcal{A}_s) \\
&\quad\quad + E(2(S(t) - S(s))S(s)|\mathcal{A}_s) \\
&\quad\quad + E(S^2(s) - s|\mathcal{A}_s) \\
&= E(S(t) - S(s)^2) - (t - s) + 2S(s)E(S(t) - S(s)) \\
&\quad\quad + (S^2(s) - s) \\
&= 0 + 0 + S^2(s) - s = S^2(s) - s \qquad \text{a.s.}
\end{aligned}$$

so that the claim holds. (Note that this shows that $\langle S\rangle(t) = t$ is the predictable variation process corresponding to the sub - martingale $S^2(t)$.) To see that $Y_t = \exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$ is a mean 1 martingale, note that $Y_t = \exp(\theta S(t) - \theta^2 t/2)$ and hence

$$\begin{aligned}
E(Y_t|\mathcal{A}_s) &= E(\exp(\theta(S(t) - S(s)))|\mathcal{A}_s) \cdot E(\exp(\theta S(s) - \theta^2 s/2)|\mathcal{A}_s) \\
&\quad\quad \cdot \exp(\theta^2(s/2 - t/2)) \\
&= E(\exp(\theta(S(t) - S(s)))) \cdot \exp(\theta^2(s/2 - t/2)) \cdot Y_s \qquad \text{a.s.} \\
&= \exp(\theta^2(t/2 - s/2)) \cdot \exp(\theta^2(s/2 - t/2)) \cdot Y_s \qquad \text{a.s.} \\
&= Y_s\,,
\end{aligned}$$

so that Y_t is a mean 1 mg. The second part of this holds simply because, with $\theta = -2\mu$ we have

$$\theta[S_\mu - \mu t] - \theta^2 t/2 = -2\mu S_\mu + 2\mu^2 t - 4\mu^2 t/2 = -2\mu S_\mu(t)\,.$$

Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a+b)$ and $E\tau = ab$.
Claim 3: If $\mu \neq 0$, then

$$P(S(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$

and

$$E(\tau) = \frac{b}{\mu} - \frac{a+b}{\mu}\,\frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}\,.$$

To prove claim 2, first consider the bounded stopping times $\tau \wedge k$. Then by the basic optional sampling theorem,

$$0 = E(S_0^2(\tau \wedge k) - \tau \wedge k)\,. \tag{3}$$

Now $\tau \wedge k \nearrow \tau$, so that $E(\tau \wedge k) \to E(\tau)$ by the monotone convergence theorem, while $S_0(\tau \wedge k) \to S_0(\tau)$ with $|S_0(\tau \wedge k)| \le a \vee b < \infty$ for all k, and hence $E(S_0^2(\tau \wedge k)) \to E(S_0^2(\tau))$ by the dominated convergence theorem. Thus taking limits across (3) yields

$$E(S_0^2(\tau)) = E(\tau)\,,$$

and when $\mu = 0$, this implies that $E(\tau) < \infty$. By playing this game with the martingale S, we find that $E(S(\tau \wedge k)) = 0$, and by the dominated convergence theorem, $E(S(\tau)) = 0$. Since $S(\tau)$ takes on the two values $-a$ and b, we have

$$0 = E_0 S(\tau) = -aP_0(S(\tau) = -a) + bP_0(S(\tau) = b) = -a(1 - p_b) + bp_b$$

so that $p_b = a/(b + a)$, $p_a = 1 - p_b = b/(b + a)$. From $E(S_0^2(\tau)) = E(\tau)$ it then follows that

$$E(\tau) = a^2 p_a + b^2 p_b = a^2 \frac{b}{b + a} + b^2 \frac{a}{b + a} = ab\,,$$

completing the proof of Claim 2.
Proof of Claim 3. Similarly, when $\mu \neq 0$, the basic optional sampling theorem yields

$$0 = E(S_\mu(\tau \wedge k) - (\tau \wedge k)\mu)\,. \tag{4}$$

Now $\tau \wedge k \nearrow \tau$, so that $E(\tau \wedge k) \to E(\tau)$ by the monotone convergence theorem, while $S_\mu(\tau \wedge k) \to S_\mu(\tau)$ with $|S_\mu(\tau \wedge k)| \le a \vee b < \infty$ for all k, and hence $E(S_\mu(\tau \wedge k)) \to E(S_\mu(\tau))$ by the dominated convergence theorem. Thus taking limits across (4) yields

$$E(S_\mu(\tau)) = \mu E(\tau)\,,$$

and this implies that $E(\tau) < \infty$ for $\mu \neq 0$. Again the basic optional sampling theorem implies that

$$E(Y(0)) = 1 = E \exp(-2\mu S_\mu(\tau \wedge k))\,,$$

for each k, and by the dominated convergence theorem this yields

$$\begin{aligned} E(Y(0)) = 1 &= E \exp(-2\mu S_\mu(\tau)) \\ &= P(S_\mu(\tau) = -a) \exp(2\mu a) + P(S_\mu(\tau) = b) \exp(-2\mu b) \\ &= p_a \exp(2\mu a) + (1 - p_a) \exp(-2\mu b) \\ &= p_a(\exp(2\mu a) - \exp(-2\mu b)) + \exp(-2\mu b) \end{aligned}$$

so that

$$p_a = \frac{1 - \exp(-2\mu b)}{\exp(2\mu a) - \exp(-2\mu b)} = \frac{1 - \exp(2\mu b)}{1 - \exp(2\mu(a+b)} \, .$$

Then, finally, since $E(S_\mu(\tau)) = \mu$,

$$\begin{aligned} E(\tau) &= \frac{E(S_\mu(\tau))}{\mu} \\ &= \frac{1}{\mu}\{-ap_a + b(1-p_a)\} \\ &= \frac{1}{\mu}\left\{ b - (a+b)\frac{1-\exp(2\mu b)}{1-\exp(2\mu(a+b)}\right\} \, . \end{aligned}$$

Note that when $\mu < 0$ we have

$$\begin{aligned} P(\|S_\mu^+\|_0^\infty \ge b) &= \lim_{a\to\infty} P(\tau_{ab} < \infty) \\ &= \lim_{a\to\infty} P(S_\mu(\tau_{ab}) = b) \\ &= \exp(-2|\mu|b) \end{aligned}$$

so that $\|S_\mu^+\|_0^\infty \sim$ Exponential$(2|\mu|)$.

5. Redo problem 3 from Problem Set #4 for yourself, not relying on the solution set, and *doing it under the assumption that* $\{S_k\}$ *is a martingale* with $E(X_k^2) < \infty$ for each k with $X_k \equiv S_k - S_{k-1}$. The X_k's need not be independent!

   Solution: This is contained in Example 13.5.1, PfS page 371, and Theorem 13.5.4, page 372. The predictable variation process for the martingale $\{S_n, \mathcal{A}_n\}_{n\ge1}$ is (with $S_0 \equiv 0$),

$$\langle S \rangle_n = \sum_{k=1}^n E\{X_k^2 | \mathcal{A}_{k-1}\}$$

   where $X_k \equiv S_k - S_{k-1}$ for $k \in \{1, \ldots, n\}$. First,

$$E(S_n^2) = \sum_{k=1}^n E\{(S_k - S_{k-1})^2\} = \sum_{k=1}^n E\{S_k^2 - S_{k-1}^2\}$$

since, by the martingale property of $\{S_k\}$,

$$E\{S_k S_{k-1}\} = E\{E[S_k S_{k-1}|\mathcal{A}_{k-1}]\} = E\{S_{k-1}E[S_k|A_{k-1}]\} = E\{S_{k-1}^2\}.$$

Thus it follows that $E|\langle S\rangle_k| \le \sum_{k=1}^n E(X_k^2) < \infty$ and

$$E|S_k^2 - \langle S\rangle_k| \le E(S_k^2) + E|\langle S\rangle_k| < \infty.$$

Furthermore, using the martingale property of $\{S_k\}$ and the definition of $\langle S\rangle_k$,

$$\begin{aligned}
E\{S_{k+1}^2 - \langle S\rangle_{k+1}|\mathcal{A}_k\} &= E\{(S_k + X_{k+1})^2 - \langle S\rangle_{k+1}|\mathcal{A}_k\} \\
&= E\{(S_k^2 + 2S_k X_{k+1} + X_{k+1}^2)|\mathcal{A}_k\} - \langle S\rangle_{k+1} \\
&= S_k^2 - \langle S\rangle_k + 2S_k E\{X_{k+1}|\mathcal{A}_k\} + E\{X_{k+1}^2|\mathcal{A}_k\} - E\{X_{k+1}^2|\mathcal{A}_k\} \\
&= S_k^2 - \langle S\rangle_k + 0 + 0 \\
&= S_k^2 - \langle S\rangle_k.
\end{aligned}$$

Thus $\{S_n^2 - \langle S\rangle_n, \mathcal{A}_n\}$ is a martingale.