

## Statistics 522, Problem Set 5 Solutions

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1. Suppose that  $\{X_n, \mathcal{A}_n\}_{n \geq 0}$  is a martingale, and assume that  $\{Y_n, \mathcal{A}_n\}_{n \geq 0}$  is a predictable process: i.e.  $Y_n$  is  $\mathcal{A}_{n-1}$  measurable for each  $n$ . Then consider the new process  $H_n \equiv \sum_{k=1}^n Y_k(X_k - X_{k-1})$ . Show that if  $\{Y_n\}$  is a bounded process, then  $\{H_n, \mathcal{A}_n\}_{n \geq 1}$  is also a martingale.

**Solution:** First note that since  $\sup_n |Y_n| \leq M < \infty$  for some  $M$ ,

$$E|H_n| \leq E\left\{\sum_{k=1}^n |Y_k| \{|X_k| + |X_{k-1}|\}\right\} \leq M \sum_{k=1}^n \{E|X_k| + E|X_{k-1}|\} < \infty.$$

Also,  $\{H_n\}$  is adapted to  $\{\mathcal{A}_n\}$  since  $\{X_n\}$  is adapted to  $\mathcal{A}_n$  and  $\{Y_n\}$  is predictable. Finally, for any  $n$ ,

$$\begin{aligned} E\{H_{n+1} | \mathcal{A}_n\} &= E\{Y_{n+1}(X_{n+1} - X_n) + \sum_{k=1}^n Y_k(X_k - X_{k-1}) | \mathcal{A}_n\} \\ &= Y_{n+1} E\{(X_{n+1} - X_n) | \mathcal{A}_n\} + H_n \quad \text{a.s.} \\ &= Y_{n+1} \cdot 0 + H_n \quad \text{a.s.} \\ &= H_n. \end{aligned}$$

Thus  $\{H_n, \mathcal{A}_n\}_{n \geq 1}$  is a martingale.

2. Exercise 12.4.1, page 313, PfS Course Notes, Chapter 12. (Exercise 12.4.1, page 306, PfS, 2000.)

Let  $T_1, T_2, \dots$  be (extended) stopping times; no ordering is assumed. Then:

- (1)  $T_1 + T_2$  is an extended stopping time if the  $\mathcal{A}_t$ 's are right-continuous.
- (2)  $A \in \mathcal{A}_{T_1}$  implies  $A \cap [T_1 \leq T_2] \in \mathcal{A}_{T_2}$ . Hint:  $[T_1 \wedge t \leq T_2 \wedge t] \in \mathcal{A}_t$ .  
 $[T_1 < T_2], [T_1 = T_2], [T_1 > T_2]$  are all in both  $\mathcal{A}_{T_1}$  and  $\mathcal{A}_{T_2}$ .
- (3)  $T_1 \leq T_2$  implies  $\mathcal{A}_{T_1} \subset \mathcal{A}_{T_2}$ . Also  $\mathcal{A}_{T_1} \cap [T_1 \leq T_2] \subset \mathcal{A}_{T_1 \wedge T_2} = \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$ .
- (4) If  $T_n \searrow T_0$  and the  $\mathcal{A}_t$ 's are right continuous, then  $\mathcal{A}_{T_0} = \bigcap_{n=1}^{\infty} \mathcal{A}_{T_n}$ .

**Solution:** (5) Let  $T_1$  and  $T_2$  be (extended) stopping times. To see that  $T_1 + T_2$  is an extended stopping time in the discrete time case (when  $T_1, T_2 \in \mathbb{N} \cup \{\infty\}$ ), note that

$$[T_1 + T_2 \leq n] = \bigcup_{k=0}^n [T_1 = k] \cap [T_2 = n - k]$$

where  $[T_1 = k] \in \mathcal{A}_k \subset \mathcal{A}_n$  and  $[T_2 = n - k] \in \mathcal{A}_{n-k} \subset \mathcal{A}_n$  for  $k = 0, \dots, n$ . Hence  $[T_1 + T_2 \leq n] \in \mathcal{A}_n$  and  $T_1 + T_2$  is an (extended) stopping time.

For the continuous case we note (from the hint) that

$$[T_1 + T_2 \leq u] = \bigcap_{m=1}^{\infty} \left\{ \bigcup_{a,b \in \mathbb{Q}, a+b \leq u} [T_1 \leq a + 1/m] \cap [T_2 \leq b + 1/m] \right\}$$

where

$$\begin{aligned} [T_1 \leq a + 1/m] &\in \mathcal{A}_{a+1/m} \subset \mathcal{A}_{a+b+2/m} \subset \mathcal{A}_{u+2/m}, \\ [T_2 \leq b + 1/m] &\in \mathcal{A}_{b+1/m} \subset \mathcal{A}_{a+b+2/m} \subset \mathcal{A}_{u+2/m}, \end{aligned}$$

for every  $m \geq 1$ . But this implies that  $[T_1 + T_2 \leq u] \in \mathcal{A}_{u+} = \mathcal{A}_u$ .

(6) Suppose  $A \in \mathcal{A}_{T_1}$ . But then

$$\begin{aligned} &A \cap [T_1 \leq T_2] \cap [T_2 \leq t] \\ &= A \cap [T_1 \leq t] \cap [T_1 \wedge t \leq T_2 \wedge t] \cap [T_2 \leq t] \\ &= \text{something in } \mathcal{A}_t \cap \text{something in } \mathcal{A}_t \cap \text{something in } \mathcal{A}_t \\ &\in \mathcal{A}_t. \end{aligned}$$

Hence  $A \cap [T_1 \leq T_2] \in \mathcal{A}_{T_2}$ .

To see that  $[T_1 < T_2]$ ,  $[T_1 = T_2]$ , and  $[T_1 > T_2]$  are all in  $\mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$ , first note that

$$\begin{aligned} &[T_1 < T_2] \cap [T_2 \leq t] \\ &= [T_1 < t] \cap [T_2 \leq t] \cap [T_1 \wedge t < T_2 \wedge t] \\ &= \bigcup_{n=1}^{\infty} [T_1 \leq t - 1/n] \cap [T_2 \leq t] \cap [T_1 \wedge t < T_2 \wedge t] \\ &= \bigcup_{n=1}^{\infty} \text{something in } \mathcal{A}_{t-1/n} \cup \text{something in } \mathcal{A}_t \cap \text{something in } \mathcal{A}_t \\ &\in \mathcal{A}_t, \end{aligned}$$

so that  $[T_1 < T_2] \in \mathcal{A}_{T_2}$ . Then by taking  $A = \Omega$  in the first part of the proof yields  $[T_1 \leq T_2] \in \mathcal{A}_{T_2}$ , and hence also  $[T_1 = T_2] = [T_1 < T_2]^c \cap [T_1 \leq T_2] \in \mathcal{A}_{T_2}$ . By symmetry we have  $[T_1 = T_2] \in \mathcal{A}_{T_1}$  as well. And by symmetry again  $[T_2 < T_1] \in \mathcal{A}_{T_1}$  and hence also  $[T_2 \leq T_1] = [T_2 < T_1] \cup [T_2 = T_1] \in \mathcal{A}_{T_1}$ . This yields  $[T_1 < T_2] = [T_2 \leq T_1]^c \in \mathcal{A}_{T_1}$ .

(7) If  $A \in \mathcal{A}_{T_1}$ , then  $A \cap [T_2 \leq t] = (A \cap [T_1 \leq t]) \cap [T_2 \leq t] \in \mathcal{A}_t$  for all  $t \geq 0$  since  $T_1 \leq T_2$ , and hence  $A \in \mathcal{A}_{T_2}$ .

Also, for  $A \in \mathcal{A}_{T_1}$ ,

$$\begin{aligned} &A \cap [T_1 \leq T_2] \cap [T_1 \wedge T_2 \leq t] \\ &= A \cap [T_1 \leq T_2] \cap ([T_1 \leq t] \cup [T_2 \leq t]) \\ &= (A \cap [T_1 \leq T_2]) \cap [T_1 \leq t] \cup (A \cap [T_1 \leq T_2]) \cap [T_2 \leq t] \\ &= \text{something in } \mathcal{A}_{T_1} \cap [T_1 \leq t] \cup \text{something in } \mathcal{A}_{T_2} \cap [T_2 \leq t] \\ &= \text{something in } \mathcal{A}_t \cup \text{something in } \mathcal{A}_t \\ &\in \mathcal{A}_t. \end{aligned}$$

for all  $t \geq 0$ , so  $\mathcal{A}_{T_1} \cap [T_1 \leq T_2] \subset \mathcal{A}_{T_1 \wedge T_2}$ .

To see that  $\mathcal{A}_{T_1 \wedge T_2} = \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$ , suppose first that  $A \in \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$ . Then

$$A \cap [T_1 \wedge T_2 \leq t] = [A \cap [T_1 \wedge T_2 \leq t] \cap ([T_1 \leq T_2] \cup [T_1 > T_2])]$$

$$\begin{aligned}
&= ((A \cap [T_1 \leq T_2]) \cap [T_1 \leq t]) \cup (A \cap [T_1 > T_2]) \cap [T_2 \leq t] \\
&= \text{something in } \mathcal{A}_{T_1} \cap [T_1 \leq t] \cup \text{something in } \mathcal{A}_{T_2} \cap [T_2 \leq t] \\
&\quad \text{by using } [T_1 \leq T_2] \in \mathcal{A}_{T_1} \text{ and } [T_1 > T_2] \in \mathcal{A}_{T_2} \\
&\in \text{something in } \mathcal{A}_t \cup \text{something in } \mathcal{A}_t \\
&\in \mathcal{A}_t.
\end{aligned}$$

For the reverse inclusion, suppose that  $A \in \mathcal{A}_{T_1 \wedge T_2}$ . But by the first part of the problem  $T_1 \wedge T_2 \leq T_j$  for  $j = 1, 2$ , and hence  $\mathcal{A}_{T_1 \wedge T_2} \subset \mathcal{A}_{T_j}$  for  $j = 1, 2$ . Thus  $A \in \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$ .

(8) Since  $T_0 \leq T_n$  for all  $n$ , it follows from (7) that  $\mathcal{A}_{T_0} \subset \mathcal{A}_{T_n}$  for all  $n$  and hence  $\mathcal{A}_{T_0} \subset \bigcap_{n=1}^{\infty} \mathcal{A}_{T_n}$ . To show the reverse inclusion, let  $A \in \bigcap_{n=1}^{\infty} \mathcal{A}_{T_n}$ . Since  $A \cap [T_n \leq t] \in \mathcal{A}_t$  for all  $t \geq 0$  and  $T_n \searrow T_0$ , it follows that  $A \cap [T_0 \leq t] \in \mathcal{A}_t$  for all  $t \geq 0$ , and hence  $A \in \mathcal{A}_{T_0}$ .

3. Exercise 13.3.6, PfS Course Notes, page 359. [Exercise 18.3.5, PfS (2000), page 477.]  
Let  $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$  be a sub-martingale with  $X_n \geq 0$ . Let  $r > 1$ . Then  $\{X_n^r\}$  is uniformly integrable if and only if  $\{X_n^r\}$  is integrable.

**Solution:** Uniform integrability implies integrability, so it remains only to prove the reverse implication. Suppose that  $\{X_n^r\}$  is integrable. Then  $\{X_n\}$  is uniformly integrable, and hence by the s-martingale convergence theorem 18.3.1,  $X_n \rightarrow X_{\infty} \in L_1$  where  $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$  is a sub-mg; i.e.  $E(X_{\infty} | \mathcal{A}_n) \geq X_n$  a.s. and

$$E(X_{\infty}^r) = E(\liminf X_n^r) \leq \liminf E(X_n^r) \leq \sup_n E(X_n^r) < \infty$$

by Fatou's lemma and integrability of  $\{X_n^r\}$ . Hence by the conditional Jensen inequality,

$$E(X_n^r) \leq E\{E(X_{\infty}^r | \mathcal{A}_n)\} \leq E\{E(X_{\infty}^r | \mathcal{A}_n)\} = E(X_{\infty}^r)$$

and it follows from Vitali's theorem that  $\{X_n^r\}$  is uniformly integrable.

Alternatively, by Doob's  $L_r$ -maximal inequality, since  $\{X_n, \mathcal{A}_n\}$  is a sub-martingale,

$$E \left\{ \left( \max_{1 \leq k \leq n} X_k \right)^r \right\} \leq \left( \frac{r}{r-1} \right)^r E|X_n|^r,$$

and hence, by the monotone convergence theorem,

$$E \left[ \sup_{1 \leq k < \infty} X_k^r \right] \leq \left( \frac{r}{r-1} \right)^r \sup_n E|X_n|^r < \infty.$$

Thus with  $Y \equiv \sup_{1 \leq k < \infty} X_k$ , it follows that

$$\sup_n E \left\{ X_n^r 1_{[X_n^r \geq \lambda]} \right\} \leq E(Y^r 1_{[Y^r \geq \lambda]}) \rightarrow 0$$

as  $\lambda \rightarrow \infty$ ; i.e.  $\{X_n^r\}$  is uniformly integrable.

4. Exercise 13.3.7, PfS Course Notes, page 359. [Exercise 18.3.7, PfS (2000), page 477.]

Let  $r > 1$ . Let  $\{X_n, \mathcal{A}_n\}_{n=0}^\infty$  be a martingale. Then the following are equivalent:

(10) The  $|X_n|^r$ -process is integrable.

(11)  $X_n \rightarrow_r X_\infty$

(12) The  $X_n$ 's are uniformly integrable (thus  $X_n \rightarrow$  (some  $X_\infty$ ) a.s.) and  $X_\infty \in L_r$ .

(13) The  $|X_n|^r$ 's are uniformly integrable.

(14)  $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^\infty$  is a submg and  $E|X_n|^r \nearrow E|X_\infty|^r$ .

**Solution:** Suppose that (10) holds. Then  $|X_n|^r$  is an integrable sub-mg. Thus the  $|X_n|^r$  are uniformly integrable by the preceding problem. Thus (13) holds.

Suppose (13) holds. Then  $\{X_n\}$  is uniformly integrable, and  $X_n \rightarrow_{a.s.} X_\infty \in L_1$  and

$$E|X_\infty|^r = E(\liminf |X_n|^r) \leq \liminf E|X_n|^r \leq \sup_n E|X_n|^r < \infty,$$

so  $X_\infty \in L_r$ ; i.e. (12) holds.

Suppose (12) holds. Then  $\{|X_n|, \mathcal{A}_n\}_{n=0}^\infty$  is a sub-martingale by Theorem 16.3.1. Thus  $|X_n| \leq E(|X_\infty| | \mathcal{A}_n)$ , so  $|X_n|^r \leq \{E(|X_\infty| | \mathcal{A}_n)\}^r \leq E(|X_n|^r | \mathcal{A}_n)$  a.s., and hence  $E|X_n|^r \leq E|X_\infty|^r < \infty$ ; i.e. (10) holds.

Thus (10) iff (12) iff (13) holds.

Now (11) implies (10) since

$$E|X_n|^r \leq c_r \{E|X_n - X_\infty|^r + E|X_\infty|^r\}$$

by the  $c_r$ -inequality.

Suppose that (13) holds. Then  $X_n \rightarrow_{a.s.} X_\infty \in L_r$  (by (13) implies (12)), and since  $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^\infty$  is a sub-mg,

$$\limsup_{n \rightarrow \infty} E|X_n|^r \leq E|X_\infty|^r < \infty.$$

Hence  $X_n \rightarrow_r X_\infty$  by Vitali's theorem; i.e. (11) holds. Thus (10) iff (12) iff (13) iff (14).

5. Exercise 13.4.3, PfS Course Notes, page 365. (Conditional Borel-Cantelli) Let  $\mathcal{A}_n$  be an increasing sequence of  $\sigma$ -fields in  $\mathcal{A}$ , and let  $A_n \in \mathcal{A}_n$ . Show that  $[A_n \text{ i.o.}] = [\omega : \sum_{n=1}^\infty P(A_n | \mathcal{A}_{n-1}) = \infty]$  almost surely.

**Solution:** To ease notation slightly, I will let the events be called  $B_n$  rather than  $A_n$ . Then we want to show that  $[B_n \text{ i.o.}] = [\omega : \sum_{k=1}^\infty P(B_k | \mathcal{A}_{k-1})(\omega) = \infty]$  almost surely. Let  $X_0 \equiv 0$  and let

$$X_n \equiv \sum_{k=1}^n \{1_{B_k} - P(B_k | \mathcal{A}_{k-1})\} = \sum_{k=1}^n 1_{B_k} - \sum_{k=1}^n P(B_k | \mathcal{A}_{k-1}) \equiv Z_n - Y_n$$

for  $n \geq 1$ . Then  $X_n$  is a martingale with bounded increments:  $|X_n - X_{n-1}| \leq 1$ ; hence  $EX_n^2 < \infty$  for all  $n \geq 0$ . With this notation our goal is to show that  $[Z_\infty = \infty] = [B_n \text{ i.o.}] = [Y_\infty = \infty]$ . Note that

$$\langle X \rangle_n = \sum_{k=1}^n E\{(\Delta X_k)^2 | \mathcal{A}_{k-1}\} = \sum_{k=1}^n E\{(1_{B_k} - P(B_k | \mathcal{A}_{k-1}))^2 | \mathcal{A}_{k-1}\} \quad (0.1)$$

$$= \sum_{k=1}^n P(B_k | \mathcal{A}_{k-1})(1 - P(B_k | \mathcal{A}_{k-1})) \quad (0.2)$$

$$\leq \sum_{k=1}^n P(B_k | \mathcal{A}_{k-1}) = Y_n \nearrow Y_\infty. \quad (0.3)$$

If  $Y_\infty(\omega) < \infty$ , then  $\langle X \rangle_\infty(\omega) < \infty$ , and hence by (a) of the martingale strengthening of the 2-series theorem proved in class (on 2/11/13),  $\lim_n X_n(\omega)$  exists and is finite, so  $Z_\infty(\omega) < \infty$ . Thus  $Z_\infty(\omega) = \infty$  implies  $Y_\infty(\omega) = \infty$ , and hence  $[Z_\infty = \infty] \subset [Y_\infty = \infty]$ . I do not have a separate proof of the reverse inclusion yet. But here is a proposition in a similar spirit to Theorem 5.3.1 of Durrett (200x), page 239, which yields the desired conclusion.

**Proposition:** Let  $\{X_n, \mathcal{A}_n\}_{n \geq 0}$  be a martingale with  $|X_{n+1} - X_n| \leq M$  almost surely. let

$$G \equiv [\lim_n X_n \text{ exists and is finite}],$$

$$H \equiv [\limsup_n X_n = +\infty, \liminf_n X_n = -\infty].$$

Then  $P(G \cup H) = 1$ .

**Proof.** Without loss of generality we can suppose  $X_0 = 0$ . Let  $K > 0$  and let  $T \equiv \inf\{n : X_n \geq K\}$ . Now  $\{X_{n \wedge T}, \mathcal{A}_{n \wedge T}\}_{n \geq 0}$  is a martingale with  $X_{n \wedge T} \leq K + M$  a.s., so  $E(X_{n \wedge T}^+) \leq E(X_{n \wedge T}) \leq K + M$ , and hence the  $s$ -martingale convergence theorem applies:  $X_{n \wedge T} \rightarrow \text{some } X_\infty \in L_1(P)$ . Thus  $\lim_n X_n$  exists on  $[T = \infty]$ . Letting  $K \rightarrow \infty$  shows that  $\lim_n X_n$  exists on  $\limsup_n X_n < \infty$ . Applying this last conclusion to  $\{-X_n\}$  shows that  $\lim_n X_n$  exists on  $\liminf_n X_n > -\infty$ . This yields the claim.  $\square$

Now we apply this to the martingale  $\{X_n, \mathcal{A}_n\}$  of the current problem:

on  $G$ ,  $Z_\infty = \infty$  if and only if  $Y_\infty = \infty$ .

on  $H$ , both  $Z_\infty = \infty$  and  $Y_\infty = \infty$ . Since  $P(G \cup H) = 1$ , the claimed result follows:  $[Z_\infty = \infty] = [Y_\infty = \infty]$  almost surely.