

Statistics 522, Problem Set 4 Solutions

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1. Exercise 13.1.4, page 353, PfS Course Notes, Chapter 13. (Exercise 18.1.4, page 471, PfS, 2000.)

Let X_1, X_2, \dots be independent rv's with each $X_k \geq 0$ and $EX_k = 1$. Let $M_n \equiv \prod_{k=1}^n X_k$ for $1 \leq k \leq n$ with $M_0 \equiv 1$. Then $\{M_n : n \geq 0\}$ is a martingale with all $E(M_n) = 1$.

Solution: Let $\mathcal{A}_n \equiv \sigma[X_1, \dots, X_n]$. Then $E|M_n| = E|\prod_{k=1}^n X_k| \leq \prod_{k=1}^n E|X_k| < \infty$. That $E(M_n) = 1$ follows easily from independence of the X_k 's and the fact that $E(X_k) = 1$ for each k . Furthermore, since X_{n+1} is independent of \mathcal{A}_n and $E(X_{n+1}) = 1$,

$$\begin{aligned} E\{M_{n+1}|\mathcal{A}_n\} &= E\left\{X_{n+1} \prod_{k=1}^n X_k | \mathcal{A}_n\right\} = E\{X_{n+1}M_n | \mathcal{A}_n\} = M_n E\{X_{n+1} | \mathcal{A}_n\} \\ &= M_n \text{ a.s.} \end{aligned}$$

Thus $\{M_n, \mathcal{A}_n\}_{n \geq 1}$ is a mean 1 martingale.

2. Exercise 13.1.6, page 353, PfS Course Notes, Chapter 13. (Exercise 18.1.6, page 471, PfS, 2000.)

Find the exponential martingale that corresponds to the martingale $\mathbb{M}(t)$ in example 1.12. Then differentiate this twice with respect to c , set $c = 0$ each time, and obtain the two martingales given in the example.

Solution: The martingale $\mathbb{M}(t)$ in example 1.12 is just $\mathbb{M}(t) = \mathbb{N}(t) - \lambda t$ where \mathbb{N} is a Poisson process. The natural filtration $\{\mathcal{A}_t\}_{t \geq 0}$ is given by $\mathcal{A}_t \equiv \sigma[\mathbb{N}(s) : s \leq t]$. To get an exponential martingale, we define

$$Y_c(t) \equiv \frac{\exp(c\mathbb{M}(t))}{E \exp(c\mathbb{M}(t))}. \quad (1)$$

Computing the denominator we find that

$$\begin{aligned} E \exp(c\mathbb{M}(t)) &= E \exp(c\mathbb{N}(t) - c\lambda t) = e^{-c\lambda t} E \exp(c\mathbb{N}(t)) = e^{-c\lambda t} \cdot e^{\lambda t(e^c - 1)} \\ &= \exp(\lambda t(e^c - 1 - c)) \end{aligned} \quad (2)$$

$$\begin{aligned} &= \exp(\lambda(t-s)(e^c - 1 - c)) \cdot \exp(\lambda s(e^c - 1 - c)) \\ &= E \exp(c(\mathbb{M}t) - \mathbb{M}(s)) \cdot E \exp(c\mathbb{M}(s)). \end{aligned} \quad (3)$$

using independence of $\mathbb{M}(t) - \mathbb{M}(s)$ and $\mathbb{M}(s)$ in the last line, since, if $N \sim \text{Poisson}(\gamma)$, the moment generating function is

$$Ee^{rN} = \sum_{k=0}^{\infty} e^{rk} e^{-\gamma} \frac{\gamma^k}{k!} = e^{-\gamma} \sum_{k=0}^{\infty} \frac{(e^r \gamma)^k}{k!} = e^{\gamma(e^r - 1)}.$$

Note that $E\{Y_c(t)\} = 1 < \infty$ for all $t > 0$. Furthermore, for $0 < s < t$, by using (3),

$$\begin{aligned} E\{Y_c(t) | \mathcal{A}_s\} &= E\left\{ \frac{\exp(c\mathbb{M}(t))}{E \exp(c\mathbb{M}(t))} \middle| \mathcal{A}_s \right\} \\ &= E\left\{ \frac{\exp(c(\mathbb{M}(t) - \mathbb{M}(s)) + c\mathbb{M}(s))}{E \exp(c\mathbb{M}(t))} \middle| \mathcal{A}_s \right\} \\ &= E\left\{ \frac{\exp(c(\mathbb{M}(t) - \mathbb{M}(s)))}{E \exp(c(\mathbb{M}(t) - \mathbb{M}(s)))} \cdot \frac{\exp(c\mathbb{M}(s))}{E \exp(c\mathbb{M}(s))} \middle| \mathcal{A}_s \right\} \\ &= \frac{\exp(c\mathbb{M}(s))}{E \exp(c\mathbb{M}(s))} \cdot E\left\{ \frac{\exp(c(\mathbb{M}(t) - \mathbb{M}(s)))}{E \exp(c(\mathbb{M}(t) - \mathbb{M}(s)))} \middle| \mathcal{A}_s \right\} \\ &= Y_c(s) \cdot 1 = Y_c(s) \quad \text{a.s.} \end{aligned}$$

since $\mathbb{M}(t) - \mathbb{M}(s)$ is independent of \mathcal{A}_s . Thus $\{Y_c(t), \mathcal{A}_t\}_{t>0}$ is a mean 1 martingale.

Using (2) in (1) we find that

$$Y_c(t) = \exp(c\mathbb{M}(t) - \lambda t(e^c - 1 - c)).$$

Differentiating Y_c with respect to c yields

$$Y'_c(t) = Y_c(t) \cdot (\mathbb{M}(t) - \lambda t(e^c - 1)).$$

Since $Y_0(t) = 1$ and $e^0 = 1$, evaluating $Y'_c(t)$ at $c = 0$ yields just $\mathbb{M}(t)$. Differentiating with respect to c again yields

$$Y''_c(t) = Y_c(t) \cdot (\mathbb{M}(t) - \lambda t(e^c - 1))^2 + Y_c(t) \cdot (-\lambda t e^c).$$

Evaluating this at $c = 0$ yields $\mathbb{M}^2(t) - \lambda t$. Thus both $\{\mathbb{M}(t), \mathcal{A}_t\}_{t \geq 0}$ and $\{\mathbb{M}^2(t) - \lambda t, \mathcal{A}_t\}_{t \geq 0}$ are martingales. This can also be verified by direct calculation.

3. Exercise 8.9.2, page 186: In the same context as Example 9.1, turn $\{S_k^2, \mathcal{A}_k\}_{1 \leq k \leq n}$ into a martingale by centering it appropriately.

Solution: If $X_k \sim (0, \sigma_k^2)$ are independent, then with $S_k \equiv X_1 + \dots + X_k$ and $\mathcal{A}_k \equiv \sigma[X_1, \dots, X_k]$, then I claim that $\{S_n^2 - \sum_{k=1}^n \sigma_k^2, \mathcal{A}_n\}_{n \geq 1}$ is a martingale. To see this we compute

$$\begin{aligned} E \left\{ S_{n+1}^2 - \sum_{k=1}^{n+1} \sigma_k^2 \middle| \mathcal{A}_n \right\} &= E \left\{ (X_{n+1} + S_n)^2 - \sigma_{n+1}^2 - \sum_{k=1}^n \sigma_k^2 \middle| \mathcal{A}_n \right\} \\ &= E \left\{ X_{n+1}^2 - \sigma_{n+1}^2 \middle| \mathcal{A}_n \right\} + S_n^2 - \sum_{k=1}^n \sigma_k^2 \\ &\quad \text{since the cross term } X_{n+1}S_n \text{ has zero conditional mean by independence} \\ &\quad \text{of } X_{n+1} \text{ and } \mathcal{A}_n, \text{ and since } E(X_{n+1}) = 0 \\ &= 0 + S_n^2 - \sum_{k=1}^n \sigma_k^2 \text{ a.s.} \end{aligned}$$

4. Exercise 8.10.1, page 189. (Exercise 8.11.1, page 249, PfS, 2000.) Show that $\{T_k, \mathcal{A}_k\}_{n \leq k \leq N}$ is a martingale and that $\text{Var}(T_N)$ is equal to the right side of (b) on page 188.

Solution: If $\{S_k, \mathcal{A}_k\}_{0 \leq k \leq N}$ is a 0-mean martingale and $X_k \equiv S_k - S_{k-1}$ has $\text{Var}(X_k) = \sigma_k^2 < \infty$ for all $1 \leq k \leq N$, then with

$$T_k \equiv S_n/b_n + \sum_{j=n+1}^k X_j/b_j, \quad \text{for } n \leq k \leq N,$$

$\{T_k, \mathcal{A}_k\}_{n \leq k \leq N}$ is a martingale. To see this, note that

$$\begin{aligned} E\{T_{k+1} | \mathcal{A}_k\} &= E\left\{ S_n/b_n + \sum_{j=n+1}^{k+1} (X_j/b_j) \middle| \mathcal{A}_k \right\} \\ &= S_n/b_n + \sum_{j=n+1}^k (X_j/b_j) + E\{(X_{k+1}/b_{k+1}) | \mathcal{A}_k\} \\ &= T_k \text{ a.s.} \end{aligned}$$

since $E(X_{k+1}|\mathcal{A}_k) = E(S_{k+1}|\mathcal{A}_k) - S_k = 0$ almost surely. Furthermore,

$$\text{Var}(T_N) = b_n^{-2} \sum_{k=1}^n \sigma_k^2 + \sum_{j=n+1}^N (\sigma_j^2/b_j^2)$$

since

$$\begin{aligned} \text{Var}(T_N) &= \text{Var}(S_n/b_n) + 2b_n^{-1} \text{Cov}(S_n, \sum_{j=n+1}^N (X_j/b_j)) + \text{Var}(\sum_{j=n+1}^N X_j/b_j) \\ &= b_n^{-2} \text{Var}(S_n) + 0 + \text{Var}(\sum_{j=n+1}^N X_j/b_j) \\ &= b_n^{-2} \sum_{k=1}^n \sigma_k^2 + \sum_{j=n+1}^N \sigma_j^2/b_j^2 \end{aligned}$$

via the following additional calculations:

$$\begin{aligned} \text{Var}(S_n) &= ES_n^2 = EE\{(S_{n-1} + X_n)^2|\mathcal{A}_{n-1}\} \\ &= E\{S_{n-1}^2 + 0 + E(X_n^2|\mathcal{A}_{n-1})\} \\ &= \text{Var}(S_{n-1}) + \text{Var}(X_n) = \text{Var}(S_{n-1}) + \sigma_n^2 \\ &= \dots = \sum_{j=1}^n \sigma_j^2, \end{aligned}$$

while, similarly,

$$\begin{aligned} \text{Var}(\sum_{j=n+1}^N X_j/b_j) &= E \left\{ \left(\sum_{n+1}^N X_j/b_j \right)^2 \right\} \\ &= EE \left\{ \left(\sum_{n+1}^{N-1} X_j/b_j \right)^2 + 2b_N^{-1} X_N \left(\sum_{n+1}^{N-1} X_j/b_j \right) + b_N^{-2} X_N^2 | \mathcal{A}_N \right\} \\ &= \text{Var} \left(\sum_{j=n+1}^{N-1} X_j/b_j \right) + 0 + b_N^{-2} \sigma_N^2 \\ &= \dots = \sum_{j=n+1}^N \sigma_j^2/b_j^2. \end{aligned}$$

5. Let Y_1, Y_2, \dots be independent random variables, and suppose that Y_k has either the density p_k or q_k with respect to some common dominating measure μ . Let $X_k \equiv q_k(Y_k)/p_k(Y_k)$ for $k \geq 1$, let P_k denote the probability measure on \mathbb{R}^∞ corresponding to p_k , and let $P = \prod_{k=1}^\infty P_k$ denote the resulting product measure on $(\mathbb{R}^\infty, \mathcal{B}^\infty)$.
- (a) Relate the X_k 's above to Kakutani's martingale as in Example 13.1.14 (PfS, page 343).
- (b) Relate the X_k 's above to the likelihood ratio martingale as in Example 13.1.13 (PfS, page 343).

Solution: (a) Note that under P then $X_k = (q_k/p_k)(Y_k)$ satisfy

$$EX_k = \int_{\mathbb{R}} \frac{q_k(y_k)}{p_k(y_k)} dP_k(y_k) = \int_{\mathbb{R}} \frac{q_k}{p_k} \cdot p_k d\mu = \int_{\mathbb{R}} q_k d\mu = 1.$$

Furthermore the X_k 's are independent with $X_k \geq 0$. Thus with $M_n \equiv \prod_{k=1}^n X_k = \prod_{k=1}^n \frac{q_k}{p_k}(Y_k)$, the $\{M_n, \mathcal{A}_n\}_{n \geq 1}$ can be viewed as Kakutani's martingale in Example 13.1.14.

(b) In the context given we can let $\mathcal{A}_n \equiv \sigma[Y_1, \dots, Y_n]$ and take (Ω, \mathcal{A}) of Example 13.1.13 to be $(\mathbb{R}^\infty, \mathcal{B}^\infty)$. Then let P_n, Q_n correspond to the densities $\prod_{k=1}^n p_k(y_k)$ and $\prod_{k=1}^n q_k(y_k)$ on $(\mathbb{R}^n, \mathcal{B}_n)$ for $n \geq 1$. With this notation, X_n of Example 13.1.14 becomes just

$$M_n = \frac{dQ_n}{dP_n} = \prod_{k=1}^n \frac{q_k}{p_k}(Y_k),$$

so that $\{M_n, \mathcal{A}_n\}_{n \geq 1}$ is a mean 1 martingale of likelihood ratios.

6. **Bonus problem:** Find the 3rd and 4th order martingales obtained by differentiating the martingale $Y \equiv Y_c$ given in Example 13.1.8 three and four times respectively and setting $c = 0$. (Hint: The Hermite polynomials defined in Exercise 12.7.3, PfS page 325, (11.6.4) page 295, and (11.6.15) page 396, might be useful.)

Solution: As we showed in class,

$$Y_c(t) = \frac{\exp(c\mathbb{B}(t))}{E \exp(c\mathbb{B}(t))} = \exp(c\mathbb{B}(t) - \frac{1}{2}c^2t)$$

is a martingale with respect to the natural filtration $\mathcal{A}_t \equiv \sigma[\mathbb{B}(s) : s \leq t]$. To find the martingales obtained by differentiating this with

respect to c and setting $c = 0$, we need to compute the derivatives of $V(x, t, c) \equiv \exp(cx - (1/2)c^2t)$ with respect to c . Thus we find

$$\begin{aligned}\frac{d}{dc}V(x, t, c) &= V(x, t, c) \cdot (x - ct), \\ \frac{d^2}{dc^2}V(x, t, c) &= V(x, t, c) \cdot (x - ct)^2 - V(x, t, c)t, \\ \frac{d^3}{dc^3}V(x, t, c) &= V(x, t, c) \cdot (x - ct)^3 - 3tV(x, t, c)(x - ct), \\ \frac{d^4}{dc^4}V(x, t, c) &= V(x, t, c) \cdot (x - ct)^4 - 6V(x, t, c)(x - ct)^2 + 3t^2V(x, t, c).\end{aligned}$$

and hence,

$$\begin{aligned}\frac{d}{dc}V(x, t, c) &\stackrel{c=0}{=} x, \\ \frac{d^2}{dc^2}V(x, t, c) &\stackrel{c=0}{=} x^2 - t, \\ \frac{d^3}{dc^3}V(x, t, c) &\stackrel{c=0}{=} x^3 - 3tx, \\ \frac{d^4}{dc^4}V(x, t, c) &\stackrel{c=0}{=} x^4 - 6tx^2 + 3t^2.\end{aligned}$$

Thus the 3rd and 4th order martingales associated with Brownian motion are:

$$H_3(\mathbb{B}(t), t) = \mathbb{B}^3(t) - 3t\mathbb{B}(t), \quad H_4(\mathbb{B}(t), t) = \mathbb{B}^4(t) - 6t\mathbb{B}^2(t) + 3t^2.$$