Statistics 522, Problem Set 2 Solutions

Wellner; 1/18/2017

1. Suppose that $\{X_k\}_{n=1}^{\infty}$ are independent and $X_k \ge 0$. Show that the following are equivalent:
(i) $\sum_{k=1}^{\infty} X_k < \infty$ almost surely;
(ii) $\sum_{k=1}^{\infty}\{P(X_k > 1) + E(X_k 1_{[X_k \le 1]})\} < \infty$;
(iii) $\sum_{k=1}^{\infty} E(X_k/(1+X_k)) < \infty$.

   Solution: Suppose that (i) holds. By the 3-series theorem it follows that all three series $I_c \equiv \sum_1^{\infty} P(|X_k| > c)$, $II_c \equiv \sum_1^{\infty} Var(X_k^{(c)}) < \infty$; and $III_c \equiv \sum_1^{\infty} E(X_k^{(c)}) < \infty$ where $X_k^{(c)} = X_k 1_{[|X_k| \le c]}$. But since the X_k's are non-negative by taking $c = 1$ this implies that $I_c + III_c = \sum_{k=1}^{\infty}\{P(X_k > 1) + E(X_k 1_{[X_k \le 1]})\} < \infty$, so (ii) holds.
Suppose that (ii) holds. Then

$$\sum_1^{\infty} Var(X_k^{(1)}) \le \sum_{k=1}^{\infty} E(X_k^{(1)})^2 \le \sum_{k=1}^{\infty} E(X_k^{(1)}) = III_{c=1} < \infty,$$

   so all three series converge with $c = 1$. Thus (i) holds by the three series theorem again.
To see that (iii) is equivalent to (ii), let $\psi(x) \equiv x 1_{[x \le 1]} + 1_{[x > 1]}$ and note that

$$\frac{1}{2}\psi(x) \le \frac{x}{1+x} \le \psi(x).$$

   Replacing x by X_k, taking expectations, and then summing on k we get

$$\frac{1}{2}\sum_{k=1}^{\infty} E\psi(X_k) \le \sum_{k=1}^{\infty} E\left(\frac{X_k}{1+X_k}\right) \le \sum_{k=1}^{\infty} E\psi(X_k).$$

   The equivalence of (iii) and (ii) follows from these inequalities.

2. Suppose that $\{Y_k\}_{k=1}^{\infty}$ are independent standard Cauchy (i.e. Cauchy$(0,1)$) random variables.
(a) Does $\sum_{k=1}^{n} 2^{-k} Y_k \to_{a.s.}$ (some rv)S?
(b) For what sequences $\{a_k\}_{k=1}^{\infty}$ does $\sum_{k=1}^{n} a_k Y_k \to_{a.s.}$ (some rv) S?

(c) What is the distribution of the limits S in (a) and (b) (if they exist)?

Solution: (a) Let $X_k \equiv 2^{-k} Y_k$. We will apply the 3-series theorem. Thus for $c > 0$ we have, since the Cauchy density is $f(y) = \pi^{-1}(1+y^2)^{-2}$ for $y \in \mathbb{R}$,

$$\begin{aligned} P(|X_k| > c) &= P(|Y_k| > 2^k c) = 2 \int_{2^k c}^{\infty} \frac{1}{\pi(1+y^2)} dy \\ &\le \frac{2}{\pi} \int_{2^k c}^{\infty} y^{-2} dy = \frac{2}{\pi c} 2^{-k}. \end{aligned}$$

Thus

$$\sum_{k=1}^{\infty} P(|X_k| > c) \le \frac{2}{\pi c} \sum_{k=1}^{\infty} 2^{-k} = \frac{2}{\pi c} < \infty.$$

Now with $X_k^{(c)} \equiv X_k 1_{[|X_k| \le c]} = 2^{-k} Y_k 1_{[|Y_k| \le 2^k c]}$ we have

$$E(X_k^{(c)}) = 2^{-k} E\left(Y_k 1_{[|Y_k| \le 2^k c]}\right) = 2^{-k} \int_{-c2^k}^{c2^k} y f(y) dy = 0$$

by symmetry, and

$$\begin{aligned} Var(X_k^{(c)} &= 2^{-2k} E\left(Y_k^2 1_{[|Y_k| \le 2^k c]}\right) = 2^{-2k} \int_{-c2^k}^{c2^k} \frac{y^2}{\pi(1+y^2)} dy \\ &\le \frac{2^{-2k}}{\pi} \cdot 2c 2^k = \frac{2c}{\pi} 2^{-k}. \end{aligned}$$

Thus $\sum_{k=1}^{n} E(X_k^{(c)}) = \sum_{k=1}^{n} 0 = 0$ and

$$\sum_{k=1}^{\infty} Var(X_k^{(c)}) \le \frac{2c}{\pi} \sum_{k=1}^{\infty} 2^{-k} = \frac{2c}{\pi} < \infty.$$

Thus by the three-series theorem it follows that

$$S_n = \sum_{k=1}^{n} X_k = \sum_{k=1}^{n} 2^{-k} Y_k \to_{a.s.} \sum_{k=1}^{\infty} 2^{-k} Y_k \equiv S.$$

(b) If 2^{-k} is replaced by a_k with $a_k \to 0$, then arguments paralleling those in (a) yield

$$P(|X_k| > c) \le \frac{2}{\pi c} a_k, \qquad E(X_k^{(c)}) = 0 \ \text{ for all } \ k \ge 1,$$

and

$$Var(X_k^{(c)}) \leq \frac{2}{\pi c} a_k,$$

so the three series all converge if $\sum_{k=1}^{\infty} a_k < \infty$. Thus the three series theorem yields $S_n \equiv \sum_{k=1}^{n} a_k Y_k \to_{a.s.} S$ for Y_k independent standard Cauchy if and only if $\sum_{k=1}^{\infty} a_k < \infty$.

(c) Note that the results of (b) make sense from the point of view of characteristic functions: since $E \exp(itY_k) = \exp(-|t|)$ for all k and $t \in \mathbb{R}$, since the Y_k's are independent we have

$$\begin{aligned} E(e^{itS_n}) &= \prod_{k=1}^{n} E e^{ita_k Y_k} = \prod_{k=1}^{n} e^{-|t| a_k} \\ &= \exp(-|t| \sum_{k=1}^{n} a_k) \to \exp(-|t| \sum_{k=1}^{\infty} a_k) \;\text{ if } A \equiv \sum_{k=1}^{\infty} a_k < \infty \\ &= E e^{itAY_1}. \end{aligned}$$

This shows that $S \stackrel{d}{=} AY_1$ where $Y_1 \sim \text{Cauchy}(0,1)$.

3. PfS, Exercise 12.3.1, page 309: Let $Z \sim N(0,1)$, let $\mathbb{V}$, $\mathbb{U}^{(1)}$, and $\mathbb{U}^{(2)}$ be independent Brownian bridge processes with Z independent of $\mathbb{V}$, $\mathbb{U}^{(j)}$, $j = 1, 2$. Fix $a > 0$. Show that:
(a) $\mathbb{B}(t) = \mathbb{V}(t) + tZ$ is a Brownian motion for $0 \leq t \leq 1$.
(b) $\mathbb{B}(at)/\sqrt{a}$, $0 \leq t < \infty$ is a Brownian motion.
(c) $\mathbb{B}(a+t) - \mathbb{B}(a)$, $t \geq 0$, is a Brownian motion.
(d) $\sqrt{1-a}\,\mathbb{U}^{(1)} \pm \sqrt{a}\,\mathbb{U}^{(2)}$ is a Brownian bridge.
(e) $\mathbb{Z}(t) \equiv \{\mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\}/\sqrt{2}$ is a Brownian bridge, $0 \leq t \leq 1/2$.

Solution: (a) It suffices to show that $\mathbb{B}$ has mean 0 and covariance $s \wedge t$ for $0 \leq s, t \leq 1$. But

$$E\mathbb{B}(t) = E\mathbb{V}(t) + tE(Z) = 0 + t \cdot 0 = 0 \;\text{ for each } t \in [0,1],$$

and, since Z is independent of $\mathbb{V}$

$$\begin{aligned} E(\mathbb{B}(s)\mathbb{B}(t)) &= E\{\mathbb{V}(s) + sZ)(\mathbb{V}(t) + tZ)\} = E\{\mathbb{V}(s)\mathbb{V}(t)\} + stE(Z^2) \\ &= s \wedge t - st + st = s \wedge t. \end{aligned}$$

(b) Note that $E\{\mathbb{B}(at)/\sqrt{a}\} = a^{-1/2}E\mathbb{B}(at) = a^{-1/2}\cdot 0 = 0$ and

$$E\left\{\frac{\mathbb{B}(as)}{\sqrt{a}}\cdot\frac{\mathbb{B}(at)}{\sqrt{a}}\right\} = \frac{1}{a}E\{\mathbb{B}(as)\mathbb{B}(at)\} = \frac{1}{a}((as)\wedge(at)) = s\wedge t.$$

(c) Let $\mathbb{W}(t) \equiv \mathbb{B}(a+t) - \mathbb{B}(a)$ for $t \geq 0$ and $a > 0$. Then $E\mathbb{W}(t) = E(\mathbb{B}(a+t) - \mathbb{B}(a)) = E\mathbb{B}(a+t) - E\mathbb{B}(a) = 0 - 0 = 0$ and

$$\begin{aligned} E\{\mathbb{W}(s)\mathbb{W}(t)\} &= E\{(\mathbb{B}(a+s) - \mathbb{B}(a))(\mathbb{B}(a+t) - \mathbb{B}(a))\} \\ &= E\{\mathbb{B}(a+s)(\mathbb{B}(a+t) - \mathbb{B}(a))\} \\ &\quad \text{ by independence of } \mathbb{B}(a+t) - \mathbb{B}(a) \text{ and } \mathbb{B}(a) \\ &= E\{\mathbb{B}(a+s)\mathbb{B}(a+t)\} - E\{\mathbb{B}(a+s)\mathbb{B}(a)\} \\ &= (a+s)\wedge(a+t) - (a+s)\wedge a \\ &= a+s-a = s \quad \text{if} \quad s \leq t; \end{aligned}$$

that is $E\{\mathbb{W}(s)\mathbb{W}(t)\} = s\wedge t$.
(d) Let $\mathbb{V}(t) \equiv \sqrt{1-a}\,\mathbb{U}^{(1)} \pm \sqrt{a}\,\mathbb{U}^{(2)}$. Then $E\mathbb{V}(t) = 0$ by linearity of expectation. By independence of $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ we find that

$$\begin{aligned} E\{\mathbb{V}(s)\mathbb{V}(t)\} &= Cov(\sqrt{1-a}\,\mathbb{U}^{(1)}(s) \pm \sqrt{a}\,\mathbb{U}^{(2)}(s), \sqrt{1-a}\,\mathbb{U}^{(1)}(t) \pm \sqrt{a}\,\mathbb{U}^{(2)}(t)) \\ &= (1-a)Cov(\mathbb{U}^{(1)}(s), \mathbb{U}^{(1)}(t)) + aCov(\mathbb{U}^{(2)}(s), \mathbb{U}^{(2)}(t)) \\ &= (1-a)(s\wedge t - st) + a(s\wedge t - st) = s\wedge t - st. \end{aligned}$$

(e) By linearity of expectation, $E\mathbb{Z}(t) = 0$. Furthermore, by independence of $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ we find that for $0 \leq s, t \leq 1/2$

$$\begin{aligned} E\{\mathbb{Z}(s)\mathbb{Z}(t)\} &= Cov(\mathbb{Z}(s), \mathbb{Z}(t)) \\ &= Cov(\mathbb{U}^{(1)}(s) + \mathbb{U}^{(2)}(1-s)\}/\sqrt{2}, \mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\}/\sqrt{2}) \\ &= 2^{-1}\{Cov(\mathbb{U}^{(1)}(s), \mathbb{U}^{(1)}(t)) + Cov(\mathbb{U}^{(2)}(s), \mathbb{U}^{(2)}(t))\} \\ &= 2^{-1}\{s\wedge t - st + (1-s)\wedge(1-t) - (1-s)(1-t)\} \\ &= 2^{-1}\{s - st + (1-t) - ((1-t) - s(1-t)\} \quad \text{if} \quad 0 \leq s \leq t \leq 1/2 \\ &= s(1-t); \end{aligned}$$

Thus $E\{\mathbb{Z}(s)\mathbb{Z}(t)\} = s\wedge t - st$ for $0 \leq s, t \leq 1/2$.

4. (a) In our proof of the existence of Brownian motion as a continuous process on $[0,1]$ we used that fact that the family of Haar functions $\{g_{nj} : \ 0 \le j \le 2^n - 1, n \ge 0\}$ is a complete orthonormal system for $L_2(0,1)$. Prove this.
(b) In our proof of the existence of Brownian motion as a continuous process on $[0,1]$ we claimed that the integrations and expectations can be interchanged in the computation of the covariance $E\{\mathbb{U}(s)\mathbb{U}(t)\}$. Justify this interchange.

Solution: (a) Recall that $g_{nj}(t) = 2^{n/2} g_{00}(2^n t - j)$ for $j \in \{0,1,2,\ldots,2^n - 1\}$ and $n \ge 1$ where $g_{00}(t) = 21_{[0,1/2]}(t) - 1$. Since g_{00} is non-zero only for $t \in [0,1]$, g_{nj} is non-zero only for $j/2^n \le t \le (j+1)/2^n$, and hence $g_{nj}(t) g_{nj'}(t) = 0$ a.e. with respect to Lebesgue measure for any $j' \ne j$. Furthermore, we compute

$$\begin{aligned} \int_0^1 g_{nj}(t) &= \int_0^1 2^{n/2} g_{00}(2^n t - j) dt = 2^{n/2} \int_{j/2^n}^{(j+1)/2^n} g_{00}(2^n t - j) dt \\ &= 2^{n/2} \int_0^{1/2^n} g_{00}(2^n s) ds = 2^{-n/2} \int_0^1 g_{00}(u) du = 2^{n/2}\{1/2 - 1/2\} = 0, \end{aligned}$$

and

$$\begin{aligned} \int_0^1 g_{nj}^2(t) &= \int_0^1 2^n g_{00}^2(2^n t - j) dt = 2^n \int_{j/2^n}^{(j+1)/2^n} g_{00}^2(2^n t - j) dt \\ &= 2^n \int_0^{1/2^n} g_{00}^2(2^n s) ds = \int_0^1 g_{00}^2(u) du = \{1/2 + 1/2\} = 1. \end{aligned}$$

Since $g_{nj}(t) g_{nj'}(t) = 0$ a.e. Lebesgue for $j' \ne j$, we have

$$\int_0^1 g_{nj}(t) g_{nj'}(t) dt = 0 \qquad \text{for} \qquad j' \ne j.$$

Furthermore if $n' \ne n$ and $j' \ne j$, we assume (without loss) that $n' > n$ and $j' > j$. Then the product $g_{n'j'}(t) g_{nj}(t) = 0$ a.e. Lebesgue unless $j/2^n \le j'/2^{n'} < (j'+1)/2^{n'} < (j+1)/2^n$, and then

$$\int_0^1 g_{n'j'}(t) g_{nj}(t) dt = (\pm 1) 2^{n'/2 + n/2} \int_0^1 g_{00}(2^{n'} t - j') 1_{[j'/2^{n'}, (j'+1)/2^{n'}]}(t) dt = 0.$$

Thus the family of Haar functions $\{g_{nj} : \ 0 \le j \le 2^n - 1, n \ge 0\}$ is orthonormal. Is it (together with the constant function 1) complete? To show this we need to show that for any $f \in L_2[0,1]$ we have, with $c_{m,j}(f) \equiv \int_0^1 f(t) g_{m,j}(t) dt$,

$$f_n(t) \equiv \sum_{m=0}^{n} \sum_{j=0}^{2^m - 1} c_{m,j}(f) g_{m,j}(t) \rightarrow_2 f(t);$$

i.e. $\int_0^1 (f_n(t) - f(t))^2 dt \to 0$. Several different proofs are possible, but I will postpone the proof for now since it will follow easily via martingale convergence theorems.

(b) Consider $\mathbb{U}(t,\omega) = \sum_{n=0}^{\infty} \mathbb{V}_n(t,\omega)$ where

$$\mathbb{V}_n(t,\omega) = \sum_{j=0}^{2^n-1} X_{n,j}(\omega) h_{n,j}(t),$$

h_{nj} are the Schauder functions, and X_{nj} are i.i.d. $N(0,1)$. By Fubini's theorem, the claim that $E\mathbb{U}(t) = 0$ will be justified if we show that

$$E\left\{ \sum_{n=0}^{\infty} |\mathbb{V}_n(t)| \right\} < \infty.$$

Now $E|X_{n,j}| = E|Z| = \sqrt{2/\pi}$ for all n and $j \in \{0, \ldots, 2^n - 1\}$. Thus the expectation in the last display is bounded by

$$\begin{aligned}
\sum_{n=0}^{\infty} E \sum_{j=0}^{2^n-1} |X_{n,j}||h_{n,j}(t)| &= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} E|X_{n,j}|h_{n,j}(t)| \\
&\le \sum_{n=0}^{\infty} \sqrt{2/\pi} 2^{-n/2} 2^{-1} = (2\pi)^{-1/2} \sum_{n=0}^{\infty} 2^{-n/2} \\
&= (2\pi)^{-1/2} \frac{1}{1 - 2^{-1/2}} = \frac{1}{\sqrt{\pi}(\sqrt{2} - 1)} < \infty.
\end{aligned}$$

It follows by Fubini's theorem that

$$E\mathbb{U}(t) = \sum_{n=0}^{\infty} E(\mathbb{V}_n(t)) = \sum_{n=0}^{\infty} 0 = 0.$$

To see that the interchanges can be made in the covariance calculation, note first that by the Cauchy-Schwarz inequality it suffices to show that $E\mathbb{U}^2(t) < \infty$ for each $t \in [0,1]$, since then we have $(E|\mathbb{U}(s)\mathbb{U}(t)|)^2 \le E\{\mathbb{U}^2(s)\}E\{\mathbb{U}^2(t)\} < \infty$. Thus it suffices to show that

$$E\left\{\left(\sum_{n=0}^{\infty}|\mathbb{V}_n(t)|\right)^2\right\} < \infty$$

But by Tonelli's theorem and independence of the $X_{n,j}$'s the left side in the last display equals

$$\begin{aligned}
&E\left\{\sum_{n=0}^{\infty}|\mathbb{V}_n(t)|\cdot\sum_{m=0}^{\infty}|\mathbb{V}_m(t)|\right\} & (1)\\
&= E\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|\mathbb{V}_n(t)||\mathbb{V}_m(t)|\right\} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}E\left\{|\mathbb{V}_n(t)||\mathbb{V}_m(t)|\right\}\\
&= \sum_{n=0}^{\infty}E|\mathbb{V}_n(t)|^2 + \sum_{m\ne n}E\left\{|\mathbb{V}_n(t)||\mathbb{V}_m(t)|\right\}\\
&\le \sum_{n=0}^{\infty}E|\mathbb{V}_n(t)|^2 + \left(\sum_{n=0}^{\infty}E\{|\mathbb{V}_n(t)|\}\right)^2 & (2)
\end{aligned}$$

where, as in our first calculation above,

$$\sum_{n=0}^{\infty}E\{|\mathbb{V}_n(t)|\} \le \frac{1}{\sqrt{\pi}(\sqrt{2}-1)} < \infty$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}E|\mathbb{V}_n(t)|^2 &= \sum_{n=0}^{\infty}E\left(\sum_{j=0}^{2^n-1}X_{nj}h_{nj}(t)\right)\left(\sum_{j'=0}^{2^n-1}X_{nj'}h_{nj'}(t)\right)\\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}h_{n,j}^2(t) \le \sum_{n=0}^{\infty}(2^{-n/2-1})^2 = 2^{-1} < \infty.
\end{aligned}$$

It follows that the right side of (2) is finite, and hence $E\mathbb{U}^2(t) < \infty$.

5. PfS, Exercise 3.2.3, page 42: Consider a measure space $(\Omega, \mathcal{A}, \mu)$. Let $\mu_0 \equiv \mu|_{\mathcal{A}_0}$ for a sub $\sigma-$field $\mathcal{A}_0$ of $\mathcal{A}$. Starting with indicator functions, show that $\int X d\mu = \int X d\mu_0$ for any $\mathcal{A}_0-$measurable function X.

Solution: (a) Suppose first that $X = 1_{D^*}$ where $D^* \in \mathcal{A}_0$. Then since $D^* \in \mathcal{A}_0$

$$\int 1_{D^*} d\mu = \mu(D^*) = \mu_0(D^*) = \int 1_{D^*} d\mu_0.$$

Thus the claimed identity holds for indicator functions.
(b) Suppose that $X = \sum_{j=1}^m a_j 1_{D_j}$ for $a_j \in \mathbb{R}$ and $D_j \in \mathcal{A}_0$ for $j = 1, \ldots, m$. Then

$$\begin{aligned}
\int X d\mu &= \int \sum_{j=1}^m a_j 1_{D_j} d\mu = \sum_{j=1}^m a_j \int 1_{D_j} d\mu \\
&= \sum_{j=1}^m a_j \int 1_{D_j} d\mu_0 \quad \text{by part (a)} \\
&= \int \sum_{j=1}^m a_j 1_{D_j} d\mu_0 \quad \text{by linearity of the integral} \\
&= \int X d\mu_0.
\end{aligned}$$

(c) If $X \geq 0$ is $\mathcal{A}_0-$measurable, then there exist simple functions $X_m \nearrow X$ which are also $\mathcal{A}_0-$measurable. Then, by the monotone convergence theorem,

$$\begin{aligned}
\int X d\mu &= \lim_m \int X_m d\mu = \lim_m \int X_m d\mu_0 \quad \text{by part (b)} \\
&= \int X d\mu_0 \quad \text{by monotone convergence again.}
\end{aligned}$$

(d) If X is a general $\mathcal{A}_0$ measurable function, then write $X = X^+ - X^-$ where X^+ and X^- are non-negative. Then by linearity of the integral and (c) it follows that

$$\begin{aligned}\int X d\mu &= \int (X^+ - X^-) d\mu = \int X^+ d\mu - \int X^- d\mu \\ &= \int X^+ d\mu_0 - \int X^- d\mu_0 \qquad \text{by (c)} \\ &= \int (X^+ - X^-) d\mu_0 = \int X d\mu_0.\end{aligned}$$