**Statistics 522, Problem Set 1 Solutions**

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1. PfS, Exercise 8.4.17, page 168:
(a) Let $X, X_{n,1}, \ldots, X_{n,n}$ be i.i.d. with the distribution function $F$. Let $F$ have finite mean $\mu = E(X)$. We know $M_n \equiv \max_{1\le k\le n} |X_{n,k}|/n \rightarrow_p 0$ by the WLLN. Trivially $E(M_n) \le E|X|$. Show that $E(M_n) = E(\max_{1\le k\le n} |X_{n,k}|/n) \rightarrow 0$.
(b) Let $X_1, X_2, \ldots$ be i.i.d. Show that $E|X| < \infty$ if and only if $M_n \rightarrow_1 0$.

**Solution:** (a) Note that $M_n \le n^{-1}\sum_{k=1}^n |X_{n,k}| \equiv Y_n$ where $Y_n \rightarrow_p E|X|$ and where $E(Y_n) = E|X|$ satisfies $\limsup_n E(Y_n) = E|X|$. Thus by Vitali's theorem $\{Y_n\}$ is uniformly integrable. Since $M_n \le Y_n$ we conclude that $\{M_n\}$ is uniformly integrable: $E(M_n 1_{[M_n \ge \lambda]}) \le E(Y_n 1_{[Y_n\ge\lambda]})$ and hence

$$\limsup_{n\to\infty} E(M_n 1_{[M_n\ge\lambda]}) \le \limsup_{n\to\infty} E(Y_n 1_{[Y_n\ge\lambda]}) \rightarrow 0$$

as $\lambda \to \infty$. Since $M_n \rightarrow_p 0$ and is uniformly integrable, Vitali's theorem yields $E(M_n) \to 0$; i.e. $M_n \rightarrow_1 0$.
(b) Now $M_n = n^{-1}\max_{1\le k\le n} |X_k|$ where the $X_k$'s are i.i.d. If $E|X_1| < \infty$, then, as in (a) $M_n \le n^{-1}\sum_{k=1}^n |X_k| \equiv Y_n$ where $Y_n \rightarrow_{a.s.} E|X|$ and where $E(Y_n) = E|X|$ satisfies $\limsup_n E(Y_n) = E|X|$. Thus by Vitali's theorem $\{Y_n\}$ is uniformly integrable. Since $M_n \le Y_n$ we conclude that $\{M_n\}$ is uniformly integrable: $E(M_n 1_{[M_n\ge\lambda]}) \le E(Y_n 1_{[Y_n\ge\lambda]})$ and hence

$$\limsup_{n\to\infty} E(M_n 1_{[M_n\ge\lambda]}) \le \limsup_{n\to\infty} E(Y_n 1_{[Y_n\ge\lambda]}) \rightarrow 0$$

as $\lambda \to \infty$. Since $M_n \rightarrow_{a.s.} 0$ and is uniformly integrable, Vitali's theorem yields $E(M_n) \to 0$; i.e. $M_n \rightarrow_1 0$.
Now suppose that $M_n \rightarrow_1 0$. Suppose that $E|X_1| = \infty$. Then

$$E \max_{1\le i\le n} \frac{|X_i|}{n} \ge E\left\{\frac{|X_n|}{n}\right\} = E\left\{\frac{|X_1|}{n}\right\} = \infty.$$

But since the left side is converging to 0, this is a contradiction. Hence $E|X_1| < \infty$.

2. PfS, Exercise 8.5.2, page 174: (Monte Carlo estimation)
Let $h : [0,1] \to [0,1]$ be continuous.
(i) Let $X_k \equiv 1\{h(\xi_k) \geq \Theta_k\}$ where $\xi_1, \xi_2, \ldots,\ \Theta_1, \Theta_2, \ldots$ are i.i.d. Uniform$(0,1)$ rv's. Show that the sample average $\overline{X}_n$ is a strongly consistent estimator of the integral $\int_0^1 h(t)dt$; i.e. $\overline{X}_n \to_{a.s.} \int_0^1 h(t)dt$.
(ii) Let $Y_k \equiv h(\xi_k)$. Show that $\overline{Y}_n \to_{a.s.} \int_0^1 h(t)dt$.
(iii) Evaluate $Var(\overline{X}_n)$ and $Var(\overline{Y}_n)$ and compare them.

**Solution:** (i) Now

$$EX_1 = E1_{[h(\xi_1)\geq\Theta_1]} = \int_0^1 \left( \int_0^1 1_{[h(v)\geq u]} du \right) dv = \int_0^1 h(v)dv < \infty$$

since $0 \leq h \leq 1$. Thus the SLLN yields $\overline{X}_n \to_{a.s.} EX_1 = \int_0^1 h(v)dv$.
(ii) Here $EY_1 = Eh(\xi_1) = \int_0^1 h(v)dv$ and the SLLN again yields $\overline{Y}_n \to_{a.s.} E(Y_1) = \int_0^1 h(t)dt$.
(iii) $Var(\overline{X}_n) = n^{-1}Var(X_1)$ and $Var(\overline{y}_n) = n^{-1}Var(y_1)$ where

$$\begin{aligned} Var(X_1) &= E(X_1^2) - (E(X_1))^2 = \int_0^1 \int_0^1 1^2_{[h(v)\geq u]} du dv - \left( \int_0^1 h(t)dt \right)^2 \\ &= \int_0^1 h(t)dt - \left( \int_0^1 h(t)dt \right)^2 \equiv \overline{h}(1-\overline{h}) \end{aligned}$$

where $\overline{h} \equiv \int_0^1 h(t)dt \in [0,1]$. Furthermore,

$$Var(Y_1) = E(Y_1^2) - (E(Y_1))^2 = \int_0^1 h^2(t)dt - \left( \int_0^1 h(t)dt \right)^2$$

where $\int_0^1 h^2(t)dt \leq \int_0^1 h(t)dt$ since $0 \leq h(t) \leq 1$ for all $\in [0,1]$. Thus $Var(Y_1) \leq Var(X_1)$ and $\overline{Y}_n$ yields an estimator of $\int_0^1 h(t)dt$ with smaller variance than $\overline{X}_n$.

3. PfS, Exercise 8.8.1, page 182: Suppose that $X_1, X_2, \ldots$ are i.i.d. with $P(X_k = 0) = P(X_k = 2) = 1/2$. Show that $S_n \equiv \sum_{k=1}^n X_k/3^k \to_{a.s.}$ some $S$, and determine the mean, variance, and the name of the d.f. $F_S$ of $S$.

**Solution:** Note that

$$\begin{aligned} E(S_n) &= \sum_{k=1}^{n} \frac{E(X_k)}{3^k} = \sum_{k=1}^{n} \frac{1}{3^k} \to \sum_{k=1}^{n} \frac{1}{3^k} = \frac{1/3}{1-1/3} = \frac{1}{2}, \\ Var(S_n) &= \sum_{k=1}^{n} \frac{Var(X_k)}{9^k} = \sum_{k=1}^{n} \frac{1}{9^k} \to \sum_{k=1}^{\infty} \frac{1}{9^k} = \frac{1/9}{1-1/9} = \frac{1}{8}. \end{aligned}$$

Thus by the two-series theorem it follows that $S_n \to S \equiv \sum_{k=1}^{\infty} X_k/3^k$. To see that $F_S(x) = P(S \le x)$ is the Cantor singular distribution $F$ given in Example 6.1.1, page 105, PfS, note that by the discussion there we know that $F$ satisfies,

$$F\left(\sum_{k=1}^{\infty} \frac{2a_k}{3^k}\right) = \sum_{k=1}^{\infty} \frac{a_k}{2^k} \quad \text{for all} \quad \{a_k\}_{k=1}^{\infty};$$

where all $a_k \in \{0,1\}$. Or, equivalently,

$$F^{-1}\left(\sum_{k=1}^{\infty} \frac{a_k}{2^k}\right) = \sum_{k=1}^{\infty} \frac{2a_k}{3^k}. \tag{1}$$

But if $U \sim \text{Uniform}[0,1]$, we may write

$$U = \sum_{k=1}^{\infty} \frac{Y_k}{2^k} \quad \text{where} \quad P(Y_k = 1) = 1/2 = P(Y_k = 0).$$

Thus by (1)

$$F^{-1}(U) = \sum_{k=1}^{\infty} \frac{2Y_k}{3^K} \stackrel{d}{=} \sum_{k=1}^{\infty} \frac{X_k}{3^k} \equiv S.$$

It follows from the inverse transformation theorem 6.3.1 that the distribution function $F_S$ of $S$ is the Cantor singular distribution function $F$.

4. PfS, Exercise 8.8.3, page 182: Suppose that $X_1, X_2, \ldots$ are arbitrary random variables with all $X_k \ge 0$ a.s. Let $c > 0$ be arbitrary. Then $\sum_{k=1}^{\infty} E(X_k \wedge c) < \infty$ implies that $\sum_{k=1}^{n} X_k \to_{a.s.}$ (some rv $S$). The converse holds for independent random variables.

**Solution:** (a) Note that:
(1) Since all $X_k \geq 0$, $\max_{n\leq m\leq N} \sum_{k=n}^m X_k \leq \sum_{k=n}^N X_k$.
(2) For any $c > 0$, $[X_k \geq c] \subset [X \wedge c \geq c]$.
From (2) it follows that

$$P(X_k \geq c) \leq P(X_k \wedge c \geq c) \leq \frac{E(X_k \wedge c)}{c},$$

and hence $\sum_{k=1}^\infty P(X_k \geq c) \leq c^{-1} \sum_{k=1}^\infty E(X_k \wedge c) < \infty$. Thus $P(X_k \geq c \ i.o.) = 0$ by the first Borel-Cantelli lemma. Thus $\{X_k\}$ and $\{X_k 1_{[X_k\leq c]} \equiv X_k^{(c)}\}$ are Khinchine equivalent, and it suffices to show that $\sum_{k=1}^n X_k^{(c)} \to_{a.s.}$ some $S^{(c)}$. But by (1) with $X_k$ replaced by $X_k^{(c)}$,

$$\begin{aligned} P(\max_{n\leq m\leq N} \sum_{k=n}^m X_k^{(c)} > \epsilon) &= P(\sum_{k=n}^N X_k^{(c)} > \epsilon) \\ &\leq \epsilon^{-1} E(\sum_{k=n}^N X_k^{(c)}) \leq \epsilon^{-1} \sum_{k=n}^N E(X_k \wedge c) \\ &\leq \epsilon^{-1} \sum_{k=n}^\infty E(X_k \wedge c) \to 0 \end{aligned}$$

as $n \to \infty$.
(b) Suppose that $S_n \equiv \sum_{k=1}^n X_k \to_{a.s.} S$ where $X_1, X_2, \ldots$ are independent. By the three-series theorem (Theorem 8.3, PfS, page 181), the three series $I_c \equiv \sum_{k=1}^\infty P(|X_k| > c)$, $II_c \equiv \sum_{k=1}^\infty Var(X_k^{(c)})$, and $III_c \equiv \sum_{k=1}^\infty EX_k^{(c)}$ all converge for every $c > 0$ where $X_k^{(c)} \equiv X_k 1_{[|X_k|\leq c]}$. But since $X_k \geq 0$,

$$X_k \wedge c = \left\{ \begin{array}{ll} X_k, & X_k \leq c \\ c, & X_k > c \end{array} \right\} = X_k^{(c)} + c1_{[X_k > c]}.$$

Taking expectations across this identity and summing on $k$ yields

$$\sum_{k=1}^\infty E(X_k \wedge c) = \sum_{k=1}^\infty \left\{ EX_k^{(c)} + cP(X_k > c) \right\} = III_c + cII_c < \infty$$

for every $c > 0$.