Statistics 522, Final Exam Solutions

Wellner; 3/16/2017

1. (30 points). **Define** *three* of the following six terms:
   (a) The conditional expectation of an integrable random variable Y conditional on $\mathcal{D}$, a sub-sigma-field of the probability space $(\Omega, \mathcal{A}, P)$.
   (b) A sub-martingale $\{X_n, \mathcal{A}_n\}$
   (c) An integrable family $\{X_n\}$ of random variables.
   (d) The predictable variation process $\langle X \rangle_n$ corresponding to a martingale $\{X_n, \mathcal{A}_n\}$ with $E(X_n^2) < \infty$ for all n.
   (e) A tight sequence of probability measures $\{P_n\}$ on the measurable space $(M, \mathcal{M})$ corresponding to a metric space (M, d) with its Borel $\sigma-$field.
   (f) Weak convergence of probability measures $\{P_n\}$ to a probability measure P on $(M, \mathcal{M})$ for a metric space (M, d).

   Solution: See course notes, PfS.

2. (30 points). Give careful **statements** of *any three* of the following five theorems or results:
   (a) The Cramér - Wold device (or method or proposition).
   (b) The simple (or basic) optional sampling theorem for an $s-$martingale $\{X_n, \mathcal{A}_n\}$.
   (c) The Lindeberg-Feller CLT.
   (d) (At least four parts of) the portmanteau theorem for convergence in distribution on a metric space (M, d).
   (e) The (classical) multivariate CLT.

   Solution: See course notes, PfS.

3. (24 points). Suppose that X and Y are random variables on the probability space $(\Omega, \mathcal{A}, P)$ with $X \in L_2(P)$ and $Y \in L_2(P)$ (so that $XY \in L_1(P)$), and suppose that $\mathcal{D}$ is a sub sigma-field of $\mathcal{A}$. Show that

$$E\{XE(Y|\mathcal{D})\} = E\{E(X|\mathcal{D})Y\} = E\{E(X|\mathcal{D})E(Y|\mathcal{D})\}\,.$$

(This can be rewritten as

$$\langle X, E(Y|\mathcal{D})\rangle = \langle E(X|\mathcal{D}), Y\rangle = \langle E(X|\mathcal{D}), E(Y|\mathcal{D})\rangle\,,$$

and thus is the "self-adjointness" property of the conditional expectation operator.)

Solution: By computing conditionally on $\mathcal{D}$ we can write

$$\begin{aligned} E\{XE(Y|\mathcal{D})\} &= E\{E[XE(Y|\mathcal{D})|\mathcal{D}]\} \\ &= E\{E(Y|\mathcal{D})E[X|\mathcal{D}]\} \\ &= E\{E[YE(X|\mathcal{D})|\mathcal{D}]\} \\ &= E\{YE(X|\mathcal{D})\}\,. \end{aligned}$$

4. (36 points).

Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $N(0,1)$ rv's. Let $S_n \equiv \sum_{k=1}^n Z_k$, and define

$$Y_n \equiv \exp(aS_n - bn)\,.$$

(a) For $r \geq 1$, prove that $Y_n \to_r 0$ if and only if $r < 2b/a^2$.

(b) When $b = a^2/2$, show that $Y_n = \prod_{j=1}^n X_j$ where the X_j's are i.i.d. with mean 1, and hence Y_n is a mean 1 martingale.

(c) Use the theory for Kakutani's martingale to show that when $b = a^2/2$ it follows that $Y_n \to_{a.s.} 0$.

Solution: (a) Now write $Y_n = \prod_{i=1}^n X_i$ where $X_i \equiv e^{aZ_i - b}$ are i.i.d. with

$$EX_i^r = E\exp(raZ_i)\exp(-rb) = e^{r^2a^2/2 - rb} \equiv \mu_r\,.$$

Note that $\mu_r < 1$ if and only if $r < 2b/a^2$. Since $Y_n^r = \prod_{i=1}^n X_i^r$, it follows by from the X_i's being i.i.d. that

$$EY_n^r = \{\mu_r\}^n \to 0$$

if and only if $r < 2b/a^2$.

(b) From (a) $EX_i = \mu_1 = e^{a^2/2 - b} = e^0 = 1$ when $b = a^2/2$. Thus $Y_n \equiv \prod_{i=1}^n X_i$ is a mean 1 martingale:

$$\begin{aligned} E(Y_{n+1}|X_1, \ldots, X_n) &= E(X_{n+1}\prod_{i=1}^n X_i|X_1, \ldots, X_n) = \prod_{i=1}^n X_i E(X_{n+1}|X_1, \ldots, X_n) \\ &= Y_n \cdot E(X_{n+1}) = Y_n \cdot 1 = Y_n \quad \text{a.s.} \end{aligned}$$

(c) To apply Kakutani's theorem we compute

$$a_n \equiv E(X_n^{1/2}) = \mu_{1/2} = \exp((1/4)a^2/2 - (1/2)a^2/2) = \exp(-a^2/8).$$

Thus $\prod_{n=1}^{N} a_n = \exp(-Na^2/8) \to 0$ as $N \to \infty$; that is, $\prod_{n=1}^{\infty} a_n = 0$. By Kakutani's theorem the martingale Y_n is not closed. Moreover $Y_n \to_{a.s.} 0$ as $n \to \infty$.
A second proof of this shows that $Y_n \to_{a.s.} 0$ if and only if $b > 0$. This proceeds via the SLLN as follows: To prove that $Y_n \to_{a.s.} 0$ if and only if $b > 0$, first suppose that $b > 0$. Then, by the SLLN,

$$a\frac{S_n}{n} - b \to_{a.s.} a \cdot 0 - b < 0\,,$$

so that $aS_n - bn \to_{a.s.} -\infty$, and hence by the continuous mapping theorem, $Y_n = \exp(aS_n - bn) \to_{a.s.} 0$.
Now suppose that $Y_n \to_{a.s.} 0$. Then we have $\log(Y_n) = aS_n - bn \to_{a.s.} -\infty$; i.e. for any large $M > 0$, there is an $N = N_M(\omega)$ such that for all ω in a set A with probability 1, there is an $N = N_M(\omega)$ so that for $n \geq N$ we have

$$aS_n(\omega) - bn \leq -M \quad \text{for} \qquad n \geq N_M(\omega)\,;$$

i.e.

$$a\frac{S_n(\omega)}{n} - b \leq -\frac{M}{n} < 0, \quad n \geq N_M(\omega)\,.$$

Since we know that $S_n/n \to_{a.s.} 0$, with probability one we can also make $aS_n(\omega)/n$ arbitrarily small for $n >$some $N'(\omega)$. Combining these two facts, it is clear that the above inequality can hold only $-b < 0$; i.e. $b > 0$. (Alternatively: since $S_n \sim N(0, n)$, if $b \leq 0$ it follows that

$$P(Y_n \geq 1) = P(aS_n - bn \geq 0) = P(aS_n \geq bn) \geq P(S_n \geq 0) = 1/2\,,$$

and hence $Y_n \not\to_{a.s.} 0$.)

Do either problem 5 or problem 6

5. (40 points). Suppose that $\{Z_n\}_{n=0}^{\infty}$ is a sequence of random variables with

$$P(Z_{n+1}=j|Z_n=i)=e^{-i}\frac{i^j}{j!}, \qquad i,j\in\{0,1,2,\ldots\}$$

with the convention that $P(X_{n+1}=0|X_n=0)=1$. Also assume that $P(Z_0=i)=1$.
(a) Interpret the conditional probability statement above in terms of Poisson random variables. Also interpret this model as a branching process. What is the off-spring distribution?
(b) Show that $\{Z_n,\mathcal{A}_n\}$ is a martingale with mean i (with respect to the filtration $\{\mathcal{A}_n\}$ with $\mathcal{A}_n=\sigma\{Z_0,\ldots,Z_n\}$ for $n\geq 0$).
(c) Give an explicit expression for $Y_n\equiv P(Z_{n+1}=0|\mathcal{A}_n)$ as a function of Z_n, and hence show that $\{Y_n,\mathcal{A}_n\}$ is a sub-martingale with $E(Y_{n+1}|\mathcal{A}_n)>Y_n$ almost surely. Use this to conclude that $Z_n\rightarrow_{a.s.} 0$.
(d) Show that for all $a>0$

$$P(\max_{0\leq n<\infty} Z_n\geq a)\leq\frac{i}{a}\,.$$

Solution: (a) Given Z_n The conditional distribution of Z_{n+1} given Z_n has a Poisson distribution with parameter $\lambda=Z_n$. Since the Poisson(λ) distribution is identical with the distribution of a sum of n independent Poisson random variables with parameter λ/n, we see that $Z_{n+1}\overset{d}{=}\sum_{j=1}^{Z_n}X_{n,j}$ where the $X_{n,j}$'s are i.i.d. Poisson(1) random variables (independent of Z_n). Thus the off-spring distribution is Poisson(1).
(b) It follows easily from (a) that $E(Z_{n+1}|Z_n)=Z_n$ a.s. since $(Z_{n+1}|Z_n)\sim$ Poisson(Z_n). Thus

$$E(Z_{n+1})=E(Z_n)=\cdots=E(Z_0)=i.$$

It follows that $\{Z_n,\mathcal{A}_n\}_{n\geq 0}$ is a mean -i martingale.
(c) It follows from (a) that

$$Y_n=P(Z_{n+1}=0|\mathcal{A}_n)=P(Z_{n+1}=0|Z_n)=\exp(-Z_n)\ \text{ a.s.}$$

Since $h(v)\equiv e^{-v}$ is convex, it follows from (b) that $\{Y_n,\mathcal{A}_n\}_{n\geq 0}$ is a sub-martingale. Since h is strictly convex and the Poisson distribution is not concentrated at its mean, $E(Y_{n+1}|\mathcal{A}_n)>Y_n$ almost surely on the event $\{Z_n>0\}$. Since $Y_n\leq 1$ (as a conditional probability), we conclude that $Y_n\nearrow 1$ almost surely. This implies that $Z_n=-\log Y_n\rightarrow_{a.s.} 0$ by continuous mapping.
(d) From Doob's maximal inequality we have

$$P(\max_{0\leq k\leq n} Z_k\geq a)\leq\frac{E(Z_n)}{a}=\frac{E(Z_0)}{a}=\frac{i}{a}.$$

Letting $n \to \infty$ and using the monotone convergence theorem it follows that

$$P(\max_{0 \le k < \infty} Z_k \ge a) \le \frac{i}{a}.$$

6. (40 points).
Let $X_1, \ldots, X_n$ be i.i.d. Poisson(1) random variables, set $S_n = X_1 + \ldots + X_n$, and $Z_n \equiv (S_n - n)/\sqrt{n}$. Prove Stirling's formula, $n! \sim \sqrt{2\pi n}(n/e)^n$, by showing that each of the following steps is valid. (As usual, $a_n \sim b_n$ for real sequences $\{a_n\}$ and $\{b_n\}$ if and only if $a_n/b_n \to 1$ as $n \to \infty$.)
(a)

$$EZ_n^- = E\left(\frac{S_n - n}{\sqrt{n}}\right)^- = \sum_{k=0}^{n} \frac{n-k}{\sqrt{n}} \cdot e^{-n} \cdot \frac{n^k}{k!} = \frac{n^{n+1/2}e^{-n}}{n!}.$$

(b) $Z_n \to_d Z \sim N(0,1)$. You may appeal to one of our theorems.
(c) Use (b) and a uniform integrability argument to show that

$$EZ_n^- \to EZ^- = 1/\sqrt{2\pi}.$$

(d) $n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n} = \sqrt{2\pi n}(n/e)^n$.

Solution: (a) First, $S_n = X_1 + \cdots + X_n \sim \text{Poisson}(n)$, and hence we can compute

$$\begin{aligned} E\left(\frac{S_n - n}{\sqrt{n}}\right)^- &= \sum_{k=0}^{n} \left(\frac{n-k}{\sqrt{n}}\right) \frac{n^k e^{-n}}{k!} \\ &= \frac{e^{-n}}{\sqrt{n}} \left\{ \frac{n}{0!} + \sum_{k=1}^{n} \left(\frac{n^{k+1}}{k!} - \frac{n^k}{(k-1)!} \right) \right\} \\ &= \frac{e^{-n}}{\sqrt{n}} \frac{n^{n+1}}{n!} \quad \text{since the sum telescopes} \\ &= \sqrt{n} e^{-n} \frac{n^n}{n!}. \end{aligned}$$

(b) Since $E(X_1) = 1$, $Var(X_1) = 1$, the Lindeberg(- Lévy) CLT yields

$$Z_n \equiv \frac{S_n - n}{\sqrt{n}} = \sqrt{n}(\overline{X}_n - 1) \to_d N(0,1).$$

(c) Note that $E(Z_n^2) = 1$ for all n, so $\{Z_n^-\}$ is uniformly integrable and it follows that

$$EZ_n^- \to EZ^- = E(Z^+) = \int_0^\infty z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}}.$$

(d) It follows from (d) and (c) that

$$\frac{n^{1/2}(n/e)^n}{n!} = E(Z_n^-) \to E(Z^-) = \frac{1}{\sqrt{2\pi}}\,.$$

This is equivalent to

$$\frac{\sqrt{2\pi n}(n/e)^n}{n!} \to 1\,,$$

i.e. $n! \sim \sqrt{2\pi n}(n/e)^n$.

Do either problem 7 or problem 8.

7. (36 points)
Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with $E(X_1) = 0$ and $E(X_1^2) < \infty$. Let $S_n = \sum_{j=1}^n X_j$, $R_n = \sum_{j=1}^n S_j$, and $\mathcal{A}_n = \sigma[X_1, \ldots, X_n]$.
(a) Show that with $Y_n \equiv R_n - nS_n$, $\{Y_n, \mathcal{A}_n\}$ is a mean 0 martingale.
(b) Now let $\{\mathbb{S}(t) : t \geq 0\}$ be standard Brownian motion on $[0, \infty)$, and let $Y(t) \equiv \mathbb{R}(t) - t\mathbb{S}(t)$ where $\mathbb{R}(t) \equiv \int_0^t \mathbb{S}(v)dv$. Show that $\{Y(t), \mathcal{A}_t\}_{t \geq 0}$ is a martingale where $\mathcal{A}_t \equiv \sigma\{\mathbb{S}(s) : \ s \leq t\}$.
(c) Now let $g(x)(t) \equiv \int_0^t x(v)dv - tx(t)$ for functions $x \in C[0,1]$. Show that g is continuous on $(C[0,1], \| \cdot \|_\infty)$; thus $g : C[0,1] \to C[0,1]$.
(d) Now consider $g(\mathbb{S}_n)$ where $\mathbb{S}_n$ is the partial sum process based on the i.i.d. X_i's at the start of the problem. Show that $g(\mathbb{S}_n) \Rightarrow$ some process $\mathbb{Y}$ where $\mathbb{Y} \in C[0,1]$ with probability 1. Identify $\mathbb{Y}$ in terms of Brownian motion $\mathbb{S}$.

Solution: (a) Now, since $Y_n = R_n - nS_n = \sum_{j=1}^n S_j - nS_n$, to show that $\{Y_n, \mathcal{A}_n\}$ is a martingale we compute

$$\begin{aligned} E\{R_{n+1} - (n+1)S_{n+1} | \mathcal{A}_n\} &= E\{R_n + S_{n+1} - S_{n+1} - nS_n - nX_{n+1} | \mathcal{A}_n\} \\ &= R_n - nS_n - nE(X_{n+1}) = R_n - nS_n = Y_n \quad \text{a.s.} \end{aligned}$$

since $E(X_{n+1}) = 0$. Thus $\{Y_n, \mathcal{A}_n\}$ is a martingale.
(b) Similarly to show that $Y(t) \equiv \mathbb{R}(t) - s\mathbb{S}(t)$ is a martingale relative to $\mathcal{A}_s \equiv \sigma[\mathbb{S}(v) : v \leq s\}$ we compute, using the martingale properties of $\{\mathbb{S}(t), \mathcal{A}_t\}$, for $0 < s \leq t$,

$$\begin{aligned} E\{\mathbb{R}(t) - t\mathbb{S}(t) | \mathcal{A}_s\} &= E\left\{\int_0^s \mathbb{S}(v)dv + \int_s^t \mathbb{S}(v)dv - t\mathbb{S}(t) | \mathcal{A}_s\right\} \\ &= \int_0^s \mathbb{S}(v)dv + E\left\{\int_s^t \mathbb{S}(v)dv | \mathcal{A}_s\right\} - t\mathbb{S}(s) \\ &= \int_0^s \mathbb{S}(v)dv - s\mathbb{S}(s) \equiv Y(s) \end{aligned}$$

where the next to last equality holds since

$$E\left\{\int_s^t \mathbb{S}(v)dv | \mathcal{A}_s\right\} = \int_s^t E\{\mathbb{S}(v) | \mathcal{A}_s\}dv = \int_s^t \mathbb{S}(s)dv = (t-s)\mathbb{S}(s) \ \text{ a.s.}$$

Thus $\{Y(t), \mathcal{A}_t\}_{t \geq 0}$ is a martingale.
(c) If $g(x)(t) \equiv \int_0^t x(v)dv - tx(t)$, then

$$\begin{aligned} \|g(x) - g(y)\|_\infty &= \sup_{0 \leq t \leq 1} | \int_0^t x(v)dv - tx(t) - (\int_0^t y(v)dv - ty(t))| \\ &\leq \sup_{0 \leq t \leq 1} \int_0^t |x(v) - y(v)|dv + \sup_{0 \leq t \leq 1} t \cdot |x(t) - y(t)| \\ &\leq \|x - y\|_\infty + 1 \cdot \|x - y\|_\infty = 2\|x - y\|_\infty. \end{aligned}$$

Thus g is a continuous (and even a Lipschitz) function from $(C[0,1], \|\cdot\|_\infty)$ to $(C[0,1], \|\cdot\|_\infty)$.

(d) Since g is continuous from (c) and the partial sum processes $\mathbb{S}_n \Rightarrow \mathbb{S}$ in $(C[0,1], \|\cdot\|_\infty)$ it follows from the continuous mapping theorem that $\mathbb{Y}_n \equiv g(\mathbb{S}_n) \Rightarrow g(\mathbb{S}) = \mathbb{Y}$ in $(C[0,1], \|\cdot\|_\infty)$ where

$$g(\mathbb{S})(t) = \int_0^t \mathbb{S}(v)dv - t\mathbb{S}(t)$$

is the martingale in (b). The process $(\mathbb{S}(t), \mathbb{Y}(t))_{t\ge 0}$ is a "singular" diffusion process in $\mathbb{R}^2$ sometimes known as *Kolmogorov's diffusion*; see e.g McKean (1963) and Groeneboom, Jongbloed, and Wellner (1999), *Ann Prob.* **27**, 1283-1303. It arises naturally in the asymptotics of nonparametric estimators of convex functions.

8. (36 points)
Suppose that $Y_1, Y_2, \ldots, Y_n, \ldots$ are i.i.d. random variables with $E(Y_1) = 0$ and $Var(Y_1) = \sigma^2$. Let $X_{n,i} \equiv a_{n,i}Y_i$ for $1 \le i \le n$ where $\{a_{n,i} : 1 \le i \le n\}$ are constants. Consider $S_n \equiv \sum_{i=1}^n X_{n,i}$.
(a) Compute $E(S_n)$ and $Var(S_n) \equiv \sigma_n^2$.
(b) Show that if $A_n^2 \equiv \max_{1\le i\le n}|a_{ni}|^2/\sum_1^n a_{ni}^2 \to 0$, then $S_n/\sigma_n \to_d Z \sim N(0,1)$.
(c) Now suppose that $a_{n,i} = n^{-1/2}(i/n)^\alpha$ for some $\alpha \in \mathbb{R}$. For what values of α does $S_n \to_d v_\alpha Z \sim N(0, v_\alpha^2)$ for some $v_\alpha^2 < \infty$? Compute v_α^2 as a function of α.

Solution: a) We compute $E(X_{ni}) = a_{ni}E(Y_i) = a_{ni}\cdot 0 = 0$ while

$$\sigma_{ni}^2 = Var(X_{ni}) = a_{ni}^2\sigma^2, \qquad \sigma_n^2 = \sum_1^n \sigma_{ni}^2 = \sigma^2\sum_1^n a_{ni}^2.$$

(b) We are in the triangular array framework of the Lindeberg-Feller CLT. It remains only to verify the Lindeberg-Feller condition. We compute

$$\begin{aligned} LF_n(\epsilon) &= \frac{1}{\sigma_n^2}\sum_{i=1}^n E\{X_{ni}^2 1_{[|X_{ni}|\ge\epsilon\sigma_n]}\} = \frac{1}{\sigma_n^2}\sum_{i=1}^n a_{ni}^2 E\{Y_i^2 1_{[|Y_i|\ge\epsilon\sigma_n/|a_{ni}|]}\} \\ &\le \frac{1}{\sigma_n^2}\sum_{i=1}^n a_{ni}^2 E\{Y_i^2 1_{[|Y_i|\ge\epsilon\sigma_n/\max_{1\le i\le n}|a_{ni}|]}\} = \frac{1}{\sigma_n^2}\sum_{i=1}^n a_{ni}^2 E\{Y_1^2 1_{[|Y_1|\ge\epsilon\sigma_n/\max_{1\le i\le n}|a_{ni}|]}\} \\ &\qquad \text{since the } Y_i's \text{ are i.i.d.} \\ &= \frac{1}{\sigma^2}E\{Y_1^2 1_{[|Y_1|\ge\epsilon\sigma_n/\max_{1\le i\le n}|a_{ni}|]}\} \to 0 \qquad \text{as } n\to\infty \end{aligned}$$

by the dominated convergence theorem since $E(Y_1^2) < \infty$ and $A_n = \max_{1\le i\le n}|a_{ni}|/\sqrt{\sum_1^n a_{ni}^2} \to 0$. It follows from the Lindeberg - Feller CLT that $S_n/\sigma_n \to Z \sim N(0,1)$ with $\sigma_n^2 = \sigma^2\sum_1^n a_{ni}^2$.

(c) When $a_{ni} = n^{-1/2}(i/n)^\alpha$ we have

$$\sum_{i=1}^n a_{ni}^2 = n^{-1}\sum_1^n (i/n)^\alpha \to \begin{cases} \int_0^1 t^{2\alpha}dt < \infty, & \text{if } \alpha > -1/2, \\ \infty, & \text{if } \alpha \le -1/2; \end{cases}$$

Note that for $\alpha > -1/2$ we then have

$$A_n^2 = \frac{\max_{1 \le i \le n} n^{-1}(i/n)^{2\alpha}}{n^{-1}\sum_1^n (i/n)^{2\alpha}} \to 0$$

since the denominator converges to $\int_0^1 t^{2\alpha} dt < \infty$ while the numerator satisfies

$$\begin{aligned} \max_{1 \le i \le n} n^{-1}(i/n)^{2\alpha} &= \begin{cases} n^{-1}, & \text{if } \ \alpha \ge 0, \\ n^{-1-2\alpha} & \end{cases} \\ &\to \ 0 \ \text{ if } \ \alpha > -1/2. \end{aligned}$$

Thus for $\alpha > -1/2$ we have

$$S_n \to_d v_\alpha Z \sim N(0, v_\alpha^2)$$

where $v_\alpha^2 = \sigma^2 \int_0^1 t^{2\alpha} dt = \sigma^2/(2\alpha + 1)$.