Statistics 522, Problem Set 9

Wellner; 3/1/2017

Reading:	
Wellner	Chapter 11, Sections 3 - 5, pages 19 - 35.
Shorack, PfS Course Notes	Sections 14.1 - 14.2, pages 395 - 408
(PfS 2000	sections 19.1-2, pages 533-544.)
Due:	Wednesday, March 8, 2017.
Reminder:	Final exam, Wednesday, March 15: 2:30-4:20.

1. (a) Let $X_1, X_2, \ldots$, be i.i.d. random variables and let $Z_n \equiv n^{-1/2}\sum_{i=1}^n X_i$. For another sequence of i.i.d. random variables $X_1', X_2', \ldots$, with each $X_i' \stackrel{d}{=} X_i$ and all X_i''s independent of the X_i's, let $X_i^s \equiv X_i - X_i'$ and set $Z_n^s \equiv n^{-1/2}\sum_{i=1}^n X_i^s$. Note that nothing has been assumed about finiteness of moments of the X_i's (or X_i's). Prove or disprove the following statement: $Z_n \to_d N(0,1)$ if and only if the symmetrized random variables $Z_n^s \to_d N(0,2)$.
(b) Now suppose that $X_1, X_2, \ldots$ are i.i.d. as in part (a), and suppose that $Z_n \equiv Z_{n,a,b} \equiv n^{1/2}(\overline{X} - a)/b$ for some $a \in \mathbb{R}$ and $b > 0$. What can you say about a and b if it is known that $Z_n \to_d N(0,1)$?

2. Suppose that $X_1, \ldots, X_m$ are i.i.d. with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$; suppose that $Y_1, \ldots, Y_n$ are i.i.d. and independent of the X_i's with $E(Y_1) = \nu$ and $Var(Y_1) = \tau^2 < \infty$.
(a) Use the classical CLT (Theorem 11.2.2, W Chapter 11) to show that $\sqrt{m}(\overline{X}_m - \mu)/\sigma \to_d N(0,1)$ as $m \to \infty$ and that $\sqrt{n}(\overline{Y}_n - \nu)/\tau \to_d N(0,1)$ as $n \to \infty$. Do these two sequences converge jointly in distribution as $m \wedge n \to \infty$?
(b) Let $N = m + n$ and set

$$D_{m,n} \equiv \sqrt{\frac{mn}{N}}\left(\overline{Y}_n - \overline{X}_m - (\nu - \mu)\right).$$

Use (a) to show that if $\lambda_N \equiv m/N \to \lambda \in [0,1]$ then

$$D_{m,n} \to_d \sqrt{\lambda}\tau Z - \sqrt{1-\lambda}\sigma Z' \sim N(0, \lambda\tau^2 + (1-\lambda)\sigma^2)$$

where $Z, Z' \sim N(0,1)$ are independent.
(c) Let $S_{m,n}^2 \equiv \lambda_N S_Y^2 + (1-\lambda_N) S_X^2$ where $S_X^2 \equiv m^{-1}\sum_{i=1}^m (X_i - \overline{X}_m)^2$ and $S_Y^2 \equiv n^{-1}\sum_{j=1}^n (Y_j - \overline{Y}_n)^2$. Show that if $\lambda_N \to \lambda \in [0,1]$ then $S_{m,n}^2 \to_p \lambda\tau^2 + (1-\lambda)\sigma^2$.
(d) Use (b) and (c) to show that if $\lambda_N \to \lambda \in [0,1]$, then $T_{m,n} \equiv D_{m,n}/S_{m,n} \to_d Z \sim N(0,1)$.
(e) Use the result of (d) and the Helly selection theorem to show that $T_{m,n} \to_d Z \sim N(0,1)$ whenever $m \wedge n \to \infty$.

3. Suppose that $\xi_1, \xi_2, \ldots$ are i.i.d. Uniform$[0,1]$ random variables. Let $\mathbb{G}_n(t) \equiv n^{-1}\sum_{i=1}^n 1_{[0,t]}(\xi_i)$ be the empirical distribution function of the ξ_i's, and let $\mathbb{U}_n(t) \equiv \sqrt{n}(\mathbb{G}_n(t) - t)$ for $0 \le t \le 1$. Use the multivariate CLT to show that all the finite-dimensional distributions of the sequence $\{\mathbb{U}_n : \ n \ge 1\}$ converge in distribution to the corresponding finite-dimensional distributions of a Brownian bridge process $\mathbb{U}$ on $[0,1]$; i.e. show that for any integer $k \ge 1$ and any points $0 < t_1 < \cdots < t_k < 1$

$$(\mathbb{U}_n(t_1), \ldots, \mathbb{U}_n(t_k)) \to_d (\mathbb{U}(t_1), \ldots, \mathbb{U}(t_k)) \sim N_k(0, (t_j \wedge t_{j'} - t_j t_{j'})_{j,j'=1}^k).$$

4. Exercise 14.1.3, PfS, page 400-401: The partial sum process $\overline{\mathbb{S}}_n$ defined in Example 11.5.2 satisfies $\overline{\mathbb{S}}_n \to_d \mathbb{S}$ in $C[0,1]$ where $\mathbb{S}$ is a standard Brownian motion process. Consider the following four functions defined on $C[0,1]$: for $x \in C[0,1]$, let (a) $g(x) \equiv \sup_{0\le t\le 1} x(t)$; (b) $g(x) \equiv \int_0^1 x(t)dt$; (c) $g(x) \equiv \lambda\{t \in [0,1] : \ x(t) > 0\}$ (where λ denotes Lebesgue measure); (d) $g(x) \equiv \inf\{t > 0 : x(t) = b\}$ with $b > 0$ fixed (where the infimum is take to be 1 if the set is empty).
For each of these real-valued functions (or "functionals" since they are real), find the discontinuity set D_g of g. Do we have $g(\mathbb{S}_n) \to_d g(\mathbb{S})$ for the g's in (a) - (d)?

5. **Optional bonus problem:** PfS, Exercise 10.1.4, page 227.
(a) Suppose the hypotheses of the classical CLT hold. Show that

$$M_n \equiv \frac{1}{\sqrt{n}} \max_{1\le k\le n} |X_{nk} - \mu| \to_p 0.$$

(b) Suppose that hypotheses of the classical Poisson Limit Theorem (see Theorem 1.2 on page 227 of PfS) hold. Show that

$$M_n \equiv \max_{1\le k\le n} |X_{nk}| \to_d \text{Bernoulli}(1 - e^{-\lambda}).$$