Statistics 522, Problem Set 6

Wellner; 2/8/2017

Reading:	
Shorack, PfS Course Notes	Chapter 13, Sections 13.6 - 13.8, pages 374 -387.
Due:	Wednesday, February 15, 2017.
Reminder 1:	Make-up lecture 2, today (8 February)
Reminder 2:	Midterm exam, Friday, February 17.

1. Polyá's urn: At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn, and is replaced together with a new ball of the same color. Just after time n, there are therefore $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n + 1)/(n + 2)$, the proportion of black balls in the urn just after time n. Prove that (relative to a natural filtration which you should specify) M_n is a martingale. Prove that $P(B_n = k) = 1/(n+1)$ for $0 \le k \le n$. What is the distribution of $\Theta \equiv \lim_n M_n$? Prove that for $0 < \theta < 1$,

$$N_n^{\theta} \equiv \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}$$

defines a martingale N_n^{θ}.

2. Suppose that $X_1, X_2, \ldots$ are independent random variables on $(\Omega, \mathcal{A})$ and that X_n has density p_n or q_n under P or Q respectively where p_n and q_n are (for simplicity) everywhere positive on $\mathbb{R}$. Let $\mathcal{F} = \sigma[X_1, X_2, \ldots]$ and $\mathcal{F}_n = \sigma[X_1, \ldots, X_n]$ for $n \ge 1$. Let $Y_n \equiv q_n(X_n)/p_n(X_n)$.
(a) Show that

$$M_n \equiv \frac{dQ}{dP}\Big|_{\mathcal{F}_n} = Y_1 \cdots Y_n$$

where the Y_n's are independent and have mean 1 under P; Hence the likelihood ratio martingale of Example 1.14 is the Kakutani product martingale of Example 1.15.
(b) Show that Q is absolutely continuous relative to P on $\mathcal{F}$ if and only

if the martingale $\{M_n, \mathcal{F}_n\}$ is uniformly integrable.
(c) Conclude from Kakutani's theorem (PfS Example 4.4, pages 482-483) that $Q \ll P$ on $\mathcal{F}$ if and only if

$$\prod_{n=1}^{\infty} E(Y_n^{1/2}) = \prod_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{p_n(x) q_n(x)} dx > 0\,.$$

(d) Construct two examples of sequences p_n and q_n, one in which the condition in (c) holds and one in which it fails. What is the statistical meaning when it holds and when it fails?

3. Let $X_1, X_2, \ldots$ be i.i.d rv's with $P(X = 1) = p$, $P(X = -1) = 1-p \equiv q$, where $0 < p < 1$ and $p \neq q$. Suppose that a, b are integers with $-a < 0 < b$. Define

$$S_n = X_1 + \ldots + X_n, \qquad T \equiv \inf\{n : S_n = -a, \text{or} \;\; S_n = b\}\,.$$

Let $\mathcal{F}_n \equiv \sigma[X_1, \ldots, X_n]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prove that $M_n \equiv (q/p)^{S_n}$ and $N_n = S_n - n(p-q)$ define martingales M_n and N_n. How would you use these martingales to deduce the values of $P(S_T = -a)$ and $E(S_T)$? [Hint: see PfS, Course Notes, pages 381-382.]

4. Exercise 13.7.2, PfS, Course Notes page 382. Suppose that S_μ is Brownian motion with drift: $S_\mu(t) = S(t) + \mu t$ for $t \geq 0$. Let $\tau_{ab} \equiv \tau \equiv \inf\{t \geq 0 : \; S_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$.
Claim 1: $S_0(t)$, $S_0^2(t) - t$, $S_\mu(t) - \mu t$ are mean 0 martingales, and, with $\theta = -2\mu$,

$$\exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$$

is a mean 1 martingale.
Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a+b)$ and $E\tau = ab$.
Claim 3: If $\mu \neq 0$, then

$$P(S_\mu(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$

and

$$E(\tau) = \frac{b}{\mu} - \frac{a+b}{\mu} \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}\,.$$

Claim 4: If $\mu < 0$, then, with $\|S_\mu^+\|_0^\infty \equiv \sup_{0<t<\infty} S_\mu(t)$,
$P(\|S_\mu^+\|_0^\infty \geq b) = \exp(-2|\mu|b)$ for all $b > 0$; i.e. $\|S_\mu^+\|_0^\infty \sim \text{Exponential}(2|\mu|)$.

(Note the analogies with problem 2.)

5. Redo problem 3 from Problem Set #4 for yourself, not relying on the solution set, and *doing it under the assumption that* $\{S_k\}$ *is a martingale* with $E(X_k^2) < \infty$ for each k with $X_k \equiv S_k - S_{k-1}$. The X_k's need not be independent!

6. **Optional bonus problem:** Exercise 13.1.4, page 353, PfS Course Notes, Chapter 13: Verify the claims made in Example 13.1.10, pages 352 and 353. In particular, verify (12), (13), (14), and (15) on pages 352-353.