Statistics 522, Problem Set 5

Wellner; 2/1/2017

Due:	Wednesday, February 8, 2017.
Reading:	Shorack, PfS (Course Notes), Chapter 13, sections 18.4 - 18.7, pages 363 - 382
Reminder 1:	Makeup lecture 1: 1 February (Wednesday), 12:30 - 1:20, Low 105.
Reminder 2:	Makeup lecture 2: 8 February (Wednesday), 12:30 - 1:20, Low 105.
Reminder 3:	Midterm Exam: Friday, February 17.

1. Suppose that $\{X_n, \mathcal{A}_n\}_{n\geq 0}$ is a martingale, and assume that $\{Y_n, \mathcal{A}_n\}_{n\geq 0}$ is a predictable process: i.e. Y_n is $\mathcal{A}_{n-1}$ measurable for each n. Then consider the new process $H_n \equiv \sum_{k=1}^n Y_k(X_k - X_{k-1})$. Show that if $\{Y_n\}$ is a bounded process, then $\{H_n, \mathcal{A}_n\}_{n\geq 1}$ is also a martingale.

2. Exercise 12.4.1, page 313, PfS Course Notes, Chapter 12. (Exercise 12.4.1, page 306, PfS, 2000.)
   Let T_1, $T_2, \ldots$ be (extended) stopping times; no ordering is assumed. Then:
   (1) $T_1 + T_2$ is an extended stopping time if the $\mathcal{A}_t$'s are right-continuous.
   (2) $A \in \mathcal{A}_{T_1}$ implies $A \cap [T_1 \leq T_2] \in \mathcal{A}_{T_2}$. Hint: $[T_1 \wedge t \leq T_2 \wedge t] \in \mathcal{A}_t$.
   $[T_1 < T_2]$, $[T_1 = T_2]$, $[T_1 > T_2]$ are all in both $\mathcal{A}_{T_1}$ and $\mathcal{A}_{T_2}$.
   (3) $T_1 \leq T_2$ implies $\mathcal{A}_{T_1} \subset \mathcal{A}_{T_2}$. Also $\mathcal{A}_{T_1} \cap [T_1 \leq T_2] \subset \mathcal{A}_{T_1 \wedge T_2} = \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$.
   (4) If $T_n \searrow T_0$ and the $\mathcal{A}_t$'s are right continuous, then $\mathcal{A}_{T_0} = \cap_{n=1}^\infty \mathcal{A}_{T_n}$.

3. Exercise 13.3.6, PfS Course Notes, page 359. [Exercise 18.3.5, PfS (2000), page 477.]
   Let $\{X_n, \mathcal{A}_n\}_{n=0}^\infty$ be a sub-martingale with $X_n \geq 0$. Let $r > 1$. Then $\{X_n^r\}$ is uniformly integrable if and only if $\{X_n^r\}$ is integrable.

4. Exercise 13.3.7, PfS Course Notes, page 359. [Exercise 18.3.7, PfS (2000), page 477.]
   Let $r > 1$. Let $\{X_n, \mathcal{A}_n\}_{n=0}^\infty$ be a martingale. Then the following are equivalent:
   (10) The $|X_n|^r-$process is integrable.
   (11) $X_n \rightarrow_r X_\infty$
   (12) The X_n's are uniformly integrable (thus $X_n \rightarrow$ (some X_∞) a.s.) and $X_\infty \in L_r$.
   (13) The $|X_n|^r$'s are uniformly integrable.
   (14) $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a submg and $E|X_n|^r \nearrow E|X_\infty|^r$.

5. Exercise 13.4.3, PfS Course Notes, page 365. (Conditional Borel-Cantelli) Let $\mathcal{A}_n$ be an increasing sequence of $\sigma-$fields in $\mathcal{A}$, and let $A_n \in \mathcal{A}_n$. Show that $[A_n\ i.o.] = [\omega : \sum_{n=1}^\infty P(A_n|\mathcal{A}_{n-1}) = \infty]$ almost surely.

6. **Bonus problem:** Exercise 13.4.4, PfS Course Notes, page 366. [Exercise 18.4.3, PfS (2000), page 484.]