Statistics 522, Problem Set 2, corrected

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Reading:

Shorack, PfS, Chapter 7, Sections 4-5, pages 130 - 146;

Shorack, PfS, Chapter 12, Section 3, pages 308 - 311.

Due: Wednesday, January 18, 2017.

1. Suppose that $\{X_k\}_{n=1}^{\infty}$ are independent and non-negative ($X_k \geq 0$). Show that the following are equivalent:
   (i) $\sum_{k=1}^{\infty} X_k < \infty$ almost surely;
   (ii) $\sum_{k=1}^{\infty} \{P(X_k > 1) + E(X_k 1_{[X_k \leq 1]})\} < \infty$;
   (iii) $\sum_{k=1}^{\infty} E(X_k/(1 + X_k)) < \infty$.

2. Suppose that $\{Y_k\}_{k=1}^{\infty}$ are independent standard Cauchy (i.e. Cauchy$(0,1)$) random variables.
   (a) Does $\sum_{k=1}^{n} 2^{-k} Y_k \rightarrow_{a.s.}$ (some rv)S?
   (b) For what sequences $\{a_k\}_{k=1}^{\infty}$ does $\sum_{k=1}^{n} a_k Y_k \rightarrow_{a.s.}$ (some rv) S?
   (c) What is the distribution of the limits S in (a) and (b) (if they exist)?

3. PfS, Exercise 12.3.1, page 309: Let $Z \sim N(0,1)$, let $\mathbb{V}$, $\mathbb{U}^{(1)}$, and $\mathbb{U}^{(2)}$ be independent Brownian bridge processes. Fix $a > 0$. Show that:
   (a) $\mathbb{B}(t) = \mathbb{V}(t) + tZ$ is a Brownian motion for $0 \leq t \leq 1$.
   (b) $\mathbb{B}(at)/\sqrt{a}$, $0 \leq t < \infty$ is a Brownian motion.
   (c) $\mathbb{B}(a + t) - \mathbb{B}(a)$, $t \geq 0$, is a Brownian motion.
   (d) $\sqrt{1 - a}\mathbb{U}^{(1)} \pm \sqrt{a}\mathbb{U}^{(2)}$ is a Brownian bridge.
   (e) $\mathbb{Z}(t) \equiv \{\mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\}/\sqrt{2}$ is a Brownian bridge, $0 \leq t \leq 1/2$.

4. (a) In our proof of the existence of Brownian motion as a continuous process on $[0,1]$ we used that fact that the family of Haar functions $\{g_{nj} : \ 0 \leq j \leq 2^n - 1, n \geq 0\}$ is a complete orthonormal system for $L_2(0,1)$. Prove this.
   (b) In our proof of the existence of Brownian motion as a continuous process on $[0,1]$ we claimed that the integrations and expectations can be interchanged in the computation of the covariance $E\{\mathbb{U}(s)\mathbb{U}(t)\}$. Justify this interchange.

5. PfS, Exercise 3.2.3, page 42: Consider a measure space $(\Omega, \mathcal{A}, \mu)$. Let $\mu_0 \equiv \mu|_{\mathcal{A}_0}$ for a sub $\sigma-$field $\mathcal{A}_0$ of $\mathcal{A}$. Starting with indicator functions, show that $\int X d\mu = \int X d\mu_0$ for any $\mathcal{A}_0-$measurable function X.

6. **Optional bonus problem 1:** PfS, Exercise 8.8.2, page 182. Suppose that $X_1, X_2, \ldots$ are independent with $X_k \sim \text{Uniform}(-k, k)$ for $k \geq 1$. (a) Show that if $0 < a < 1$, then $S_n \equiv \sum_{k=1}^{n} a^k X_k \to_{a.s.}$ (some rv) S. (b) Evaluate the mean and the variance of S (give simple expressions in terms of a).