Statistics 522, Problem Set 1

Wellner; 1/04/2017

Reading: Shorack, PfS, Chapter 8, Sections 4-8, pages 162 - 185; Durrett, Probability, Section 2.5, pages 78 - 86.

Due: Wednesday, January 11, 2017.

1. PfS, Exercise 8.4.17, page 168:
   (a) Let $X, X_{n,1}, \ldots, X_{n,n}$ be i.i.d. with the distribution function F. Let F have finite mean $\mu = E(X)$. We know $M_n \equiv \max_{1 \le k \le n} |X_{n,k}|/n \rightarrow_p 0$ by the WLLN. Trivially $E(M_n) \le E|X|$. Show that $E(M_n) = E(\max_{1 \le k \le n} |X_{n,k}|/n) \rightarrow 0$.
   (b) Let $X_1, X_2, \ldots$ be i.i.d. Show that $E|X| < \infty$ if and only if $M_n \rightarrow_1 0$.

2. PfS, Exercise 8.5.2, page 174: (Monte Carlo estimation)
   Let $h : [0,1] \rightarrow [0,1]$ be continuous.
   (i) Let $X_k \equiv 1\{h(\xi_k) \ge \Theta_k\}$ where $\xi_1, \xi_2, \ldots, \Theta_1, \Theta_2, \ldots$ are i.i.d. Uniform$(0,1)$ rv's. Show that the sample average $\overline{X}_n$ is a strongly consistent estimator of the integral $\int_0^1 h(t)dt$; i.e. $\overline{X}_n \rightarrow_{a.s.} \int_0^1 h(t)dt$.
   (ii) Let $Y_k \equiv h(\xi_k)$. Show that $\overline{Y}_n \rightarrow_{a.s.} \int_0^1 h(t)dt$.
   (iii) Evaluate $Var(\overline{X}_n)$ and $Var(\overline{Y}_n)$ and compare them.

3. PfS, Exercise 8.8.1, page 182: Suppose that $X_1, X_2, \ldots$ are i.i.d. with $P(X_k = 0) = P(X_k = 2) = 1/2$. Show that $\sum_{k=1}^n X_k/3^k \rightarrow_{a.s.}$ some S, and determine the mean, variance, and the name of the d.f. F_S of S.

4. PfS, Exercise 8.8.3, page 182: Suppose that $X_1, X_2, \ldots$ are arbitrary random variables with all $X_k \ge 0$ a.s. Let $c > 0$ be arbitrary. Then $\sum_{k=1}^\infty E(X_k \wedge c) < \infty$ implies that $\sum_{k=1}^n X_k \rightarrow_{a.s.}$ (some rv S). The converse holds for independent random variables.

5. **Optional bonus problem 1:** The Marcinkiewicz - Zygmund strong law of large numbers is as follows: Suppose that $X_1, X_2, \ldots$ are i.i.d. with $E(X_1) = 0$ and $E|X_1|^p < \infty$ for some $1 < p < 2$. If $S_n \equiv X_1 + \cdots + X_n$, then $S_n/n^{1/p} \rightarrow_{a.s.} 0$. Prove that the converse also holds: if $S_n/n^{1/p} \rightarrow_{a.s.} 0$ for some $p > 0$ then $E|X_1|^p < \infty$.

6. **Optional bonus problem 2:** (From Durrett, problem 2.5.3, page 85.) Let $X_1, X_2, \ldots$ be i.i.d. standard normals. Show that for any t the series $\sum_{n=1}^{\infty} X_n \cdot n^{-1} \cdot \sin(n\pi t)$ converges almost surely.