Statistics 522, Problem Set 8 Solutions

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Reminder: Final exam, Wednesday, March 20.

1. PfS Course Notes, Exercise 9.3.5; PfS (2000), Exercise 13.1.4, page 371. Show that the real part of a characteristic function (or $Re\phi(\cdot)$) is itself a characteristic function.

   Solution: Note that since $\overline{\phi}_X = \overline{\phi}_{-X}$ we have

$$
\begin{aligned}
\text{Re}\phi_X(t) &= \frac{1}{2}(\phi_X(t) + \overline{\phi}_X(t)) = \frac{1}{2}(\phi_X(t) + \phi_{-X}(t)) \\
&= \frac{1}{2}\left(Ee^{itX} + Ee^{-itX}\right) = \frac{1}{2}\left(\int e^{itx} dF_X(x) + \int e^{itx} dF_{-X}(x)\right) \\
&= \int e^{itx} d(F_X(x) + F_{-X}(x))/2 \equiv \int e^{itx} dG(x)
\end{aligned}
$$

   where $G(x) \equiv (1/2)(F_X(x) + F_{-X}(x))$ is the distribution function of ϵX where ϵ is a Rademacher random variable independent of X.

2. PfS Course Notes, Exercise 9.3.6; PfS (2000), Exercise 13.1.5, page 371. Let ϕ be a chf. Show that $c^{-1}\int_0^c \phi(tu)du$ is a chf.

   Solution: Let U be a Uniform$(0, c)$ random variable independent of X. Then let $Y \equiv UX$. The characteristic function of Y is

$$
\begin{aligned}
\phi_Y(t) &= Ee^{itY} = Ee^{itUX} = E\{E\{e^{itUX}|U\}\} \\
&= E\{\phi_X(tU)\} = \frac{1}{c}\int_0^c \phi_X(tu)du.
\end{aligned}
$$

   Thus if ϕ is the characteristic function of X, then the given expression is the characteristic function of UX where $U \sim$ Uniform$(0, c)$ is independent of X.

   In fact a random variable Y has a unimodal distribution if and only if it has a characteristic function of the form given in the display with $c = 1$ (and hence also if and only if $Y = UX$ for $U \sim$ Uniform$(0, 1)$ and $X \sim F$); see e.g. Dharmadhikari,and Joag-Dev (1988), *Unimodality, Convexity, and Applications*, page 7.

3. PfS Course Notes, Exercise 9.4.2; PfS (2000), Exercise 13.2.2, page 348.

Solution: If $X, X' \sim \text{Uniform}(0,1)$ are independent, then by Proposition 3.1(c) and the chf of Uniform(0, 1) given on page 203, with $T \equiv X - X'$ we have

$$\begin{aligned}\phi_T(t) &= Ee^{itT} = Ee^{it(X-X')} = \phi_X(t)\overline{\phi}_X(t) = \\ &= \frac{(e^{it}-1)}{it}\frac{(e^{-it}-1)}{-it} = \frac{1-(e^{it}+e^{-it})+1}{t^2} = 2(1-\cos t)/t^2,\end{aligned}$$

which is the claimed characteristic function of the triangular density as claimed in the table on page 203. (Note that this shows that $T = X - X'$ has the claimed triangular density via the inversion formula given in Theorem 9.4.3 since $\int |\phi_T(t)|dt < \infty$: $|\phi_T(t)| \le 4/t^2$ for $|t| \ge 1$ while $|\phi_T(t)| \le 1$ for $|t| \le 1$.)

Now the de Vallée-Poussin density is given by $f_D(x) = (1-\cos(x))/(\pi x^2)$ on $\mathbb{R}$, and hence $f_D(x) = \phi_T(x)/(2\pi)$ where ϕ_T is the characteristic function of the triangular density that we just derived. Hence by Theorem 4.3 we have

$$\begin{aligned}\phi_D(t) &= Ee^{itD} = \int e^{itx} f_D(x)dx = \int e^{itx}\phi_T(x)/(2\pi)dx \\ &= \int e^{-ity}\phi_T(y)/(2\pi)dy \quad \text{using} \quad \phi_T(-y) = \phi_T(y) \\ &= f_T(t) \quad \text{by Theorem 4.3} \\ &= (1-|t|)1_{[-1,1]}(t).\end{aligned}$$

If $X \sim f_D$, the de la Vallée-Poussin density, then $\phi_X(t)$ is triangular, and we compute $\phi'_X(t) = +1$ for $t < 0$ while $\phi'_X(t) = -1$ for $t > 0$. Thus $\phi'_X(t)$ is discontinuous at $t = 0$, and it follows from Exercise 9.6.1 that $E|X| = \infty$. This also follows by direct computation from the density:

$$E|X| = \frac{2}{\pi}\int_0^\infty x\frac{1-\cos(x)}{x^2}dx = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(x)}{x}dx$$

where $1 - \cos(x) \ge 1$ for $\pi/2 \le x \le 3\pi/2$ since $\cos(x)$ is negative in this interval. By periodicity of $\cos(x)$, this lower bound is also valid for the intervals $[(4k+1)\pi/2, (4k+3)\pi/2]$, $k = 0, 1, 2, \ldots$. It follows that

$$\begin{aligned}\int_0^\infty \frac{1-\cos(x)}{x} dx &\geq \sum_{k=0}^\infty \int_{(4k+1)\pi/2}^{(4k+3)\pi/2} x^{-1} dx \\ &\geq \sum_{k=0}^\infty \frac{1}{(4k+3)\pi/2} \left((4k+3) - (4k+2)\right)\pi/2 \\ &= \sum_{k=0}^\infty \frac{2}{4k+3} \geq \frac{1}{2} \sum_{k=0}^\infty \frac{1}{k+1} = \infty.\end{aligned}$$

Thus $E|X| = \infty$.

4. (a) Let $X_1, X_2, \ldots,$ be i.i.d. random variables and let $Z_n \equiv n^{-1/2}\sum_{i=1}^n X_i$. For another sequence of i.i.d. random variables $X_1', X_2', \ldots,$ with each $X_i' \stackrel{d}{=} X_i$ and all X_i''s independent of the X_i's, let $X_i^s \equiv X_i - X_i^s$ and set $Z_n^s \equiv n^{-1/2}\sum_{i=1}^n X_i^s$. Note that nothing has been assumed about finiteness of moments of the X_i's (or X_i's). Prove or disprove the following statement: $\overline{Z}_n \to_d N(0,1)$ if and only if the symmetrized random variables $\overline{Z}_n^s \to_d N(0,2)$.
(b) Now suppose that $X_1, X_2, \ldots$ are i.i.d. as in part (a), and suppose that $Z_n \equiv Z_{n,a,b} \equiv n^{1/2}(\overline{X} - a)/b$ for some $a \in \mathbb{R}$ and $b > 0$. What can you say about a and b if it is known that $Z_n \to_d N(0,1)$?

Solution: (a) Suppose that $Z_n \to_d N(0,1)$. Then it follows (from e.g. Exercise 9.1.3) that $\{Z_n\}$ is tight, and thus from the converse classical CLT (Theorem 10.7.1, PfS course notes page 265) as proved in class that $E(X_1) = 0$ and $E(X_1^2) < \infty$. Hence $E(X_1 - X_1') = 0$ and $E(X_1 - X_1')^2 = EX_1^2 + E(X_1')^2 = 2$. Then by the classical CLT we have $(Z_n, Z_n') \to_d (Z, Z')$ where Z, Z' are independent. Thus we have

$$Z_n^s \ = Z_n - Z_n' \to_d Z - Z' \sim N(0,2).$$

Now Suppose that $Z_n^s \to_d N(0,2)$. Then $\{Z_n^s\}$ is tight, and by the converse to the classical CLT we have $0 = E(X_1 - X_1')$ and $E(X_1 - X_1')^2 < \infty$. By independence of the X_i's together with their being identically distributed this implies $2EX_1^2 < \infty$ and hence $EX_1^2 < \infty$. This entails that $E|X_1| < \infty$ and hence that $\mu \equiv E(X_1) = E(X_1')$ is well-defined. This does *not* imply that $E(X_1) = 0$. But it does imply that $W_n \equiv n^{1/2}(\overline{X}_n - \mu) = n^{-1/2}\sum_{i=1}^n (X_i - \mu)$ satisfies $W_n \to_d Z \sim N(0,1)$. Thus the two statements are not quite equivalent, but they are equivalent up to a centering of Z_n.

(b) If we know that $Z_n(a,b) \to_d N(0,1)$ for some constants a, b with $b > 0$, then $\{Z_n(a,b)\}$ is tight. Since

$$Z_n(a,b) = n^{-1/2}\sum_{i=1}^n (X_i - a)/b \equiv n^{-1/2}\sum_{i=1}^n Y_i,$$

we know from the converse CLT that $E(Y_1)^2 = E(X_1 - a)^2/b^2 < \infty$ and that $0 = E(Y_1) = E(X_1 - a)/b$. This yields $a = E(X_1) = \mu$, and hence $EY_1^2 = E(X_1 - \mu)^2/b^2 = \sigma^2/b^2$. But then by the classical CLT it follows that $Z_n(a,b) \to_d N(0, \sigma^2/b^2)$. But by our hypothesis $Z_n(a,b) \to_d N(0,1)$, and thus $\sigma^2/b^2 = 1$, or $\sigma^2 = b^2$.

5. PfS Course Notes, Exercise 10.1.1, page 226; PfS (2000), Exercise 14.1.1, page 366.
For each $n \geq 1$, let $X_{n1}, \ldots, X_{nn}$ be i.i.d. with finite mean μ. Use characteristic functions to show the WLLN result that $\overline{X}_n \to_p \mu$ as $n \to \infty$. Equivalently, show that

$$\overline{X}_n \to_d \delta_\mu \equiv \text{the degenerate distribution with mass 1 at } \mu.$$

Remark: an additional hypothesis seems to be needed here, namely $\limsup_{n\to\infty} E|X_{n1}| < \infty$.

Solution: Since $E|X_1| < \infty$, it follows from Inequality 9.6.2, PfS Course Notes page 213, that

$$\left|\phi_{X_1}(t) - (1 + (it)EX_1)\right| \leq 3|t|E\{|X_{n1}|\}g(t) \quad \text{for all} \quad t \in \mathbb{R}.$$

where $g(t) \to 0$ as $t \to 0$. Since $\phi_{\overline{X}_n}(t) = \phi_{X_{n1}}(t/n)^n$ it follows from the product inequality Lemma 6.4 (page 213) together with the inequality of the last display that

$$\begin{aligned}
&\left|\phi_{\overline{X}_n}(t) - (1 + i(t/n)E(X_1))^n\right| \\
&\quad = \left|\phi_{X_{n1}}(t/n)^n - (1 + i(t/n)\mu)^n\right| \\
&\quad \leq 3n|t/n|E\{|X_{n1}|\}g(t/n) = 3|t|E\{|X_{n1}|\}g(t/n) \to 0
\end{aligned}$$

as $n \to \infty$ if $\limsup_{n\to\infty} E|X_{n1}| < \infty$. Since $(1+(it\mu)/n)^n \to \exp(it\mu)$ as $n \to \infty$, it follows that $\phi_{\overline{X}_n}(t) \to \exp(it\mu)$. This implies that $\overline{X}_n \to_d \mu$. Since convergence in distribution to a constant implies convergence in probability to the same constant, we conclude that $\overline{X}_n \to_p \mu$.