Statistics 522, Problem Set 7 Solutions

Wellner; 2/27/2013

1. Exercise 9.2.4, PfS Course Notes, page 199. (Exercise 11.8.4, page 293, PfS 2000).

   Solution: Now by changing variables to $y = \log x$ we get

$$
\begin{aligned}
EX^k &= \int_0^\infty x^k x^{-1} (2\pi)^{-1/2} \exp(-(\log x)^2/2) dx \\
&= \int_{-\infty}^\infty e^{ky} \phi(y) dy \quad \text{where} \quad \phi(z) = (2\pi)^{-1/2} e^{-z^2/2} \\
&= e^{k^2/2} \int_{-\infty}^\infty \phi(z-k) dz = e^{k^2/2}.
\end{aligned}
$$

   On the other hand, by the same change of variables,

$$
\begin{aligned}
EY_a^k &= \int_0^\infty x^k x^{-1} (2\pi)^{-1/2} \exp(-(\log x)^2/2) \left(1 + a \sin(2\pi \log x)\right) dx \\
&= \int_{-\infty}^\infty e^{ky} \phi(y) \left(1 + a \sin(2\pi y)\right) dy \\
&= e^{k^2/2} \int_{-\infty}^\infty \phi(y-k) \left(1 + a \sin(2\pi y)\right) dy \\
&= e^{k^2/2} \int_{-\infty}^\infty \phi(z) \left(1 + a \sin(2\pi (z+k))\right) dz \\
&= e^{k^2/2} \int_{-\infty}^\infty \phi(z) \left(1 + a \sin(2\pi z)\right) dz \\
&\qquad \text{by using} \\
&\qquad \sin(2\pi(z+k)) = \sin(2\pi z)\cos(2\pi k) + \cos(2\pi z)\sin(2\pi k) \\
&\qquad\qquad = \sin(2\pi z) \cdot 1 + \cos(2\pi z) \cdot 0 \\
&\qquad\qquad = \sin(2\pi z) \\
&= e^{k^2} \left(1 + a \int_{-\infty}^\infty \sin(2\pi z) \phi(z) dz\right) \\
&= e^{k^2/2} \quad \text{since } \sin \text{ is odd and } \phi \text{ is even} \\
&= e^{k^2/2}.
\end{aligned}
$$

2. Exercise 11.6.6, page 34, Wellner, Chapter 11, notes.

   Solution: *Proposition 2.1 in* $\mathbb{R}^k$*:* Suppose that $\{X_n\}$, X are random vectors in $\mathbb{R}^k$ and suppose that $Ef(X_n) \to Ef(X)$ for each $f \in C^\infty(\mathbb{R})$, the class of all bounded functions on $\mathbb{R}^k$ with bounded derivatives of all orders. Then $X_n \to_d X$.

   Proof. Let $X \sim N_k(0, I)$. For a fixed $f \in BL(\mathbb{R}^k)$ and $\sigma > 0$, define a smoothed function $f_\sigma : \mathbb{R}^k \to \mathbb{R}$ as follows:

$$f_\sigma(x) = Ef(x + \sigma Z) = \frac{1}{(2\pi\sigma^2)^{k/2}} \int \cdots \int f(y) \exp\left(-\frac{|x-y|^2}{2\sigma^2}\right) dy.$$

   Then $f_\sigma \in C^\infty(\mathbb{R}^k)$ since we can justify repeated differentiation by the dominated convergence theorem. Furthermore

$$\begin{aligned} \sup_{x\in\mathbb{R}^k} |f_\sigma(x) - f(x)| &\leq \sup_{x\in\mathbb{R}^k} E|f(x+\sigma Z) - f(x)| \\ &\leq \|f\|_{BL} E\{1 \wedge \sigma|Z|\} \end{aligned}$$

   where the right side converges to zero as $\sigma \to 0$ by the dominated convergence theorem.

   As in the case of $\mathbb{R}$, for fixed $\epsilon > 0$ we can choose $\sigma > 0$ so that $\|f_\sigma - f\|_\infty \leq \epsilon$. Then

$$|Ef(X_n) - Ef(X)| \leq |Ef_\sigma(X_n) - Ef_\sigma(X)| + 2\epsilon$$

   so that

$$\limsup_{n\to\infty} |Ef(X_n) - Ef(X)| \leq 2\epsilon$$

   since $f_\sigma \in C^\infty(\mathbb{R}^k)$ and hence $Ef_\sigma(X_n) \to Ef_\sigma(X)$ by our hypothesis. Thus $X_n \to_d X$ by the portmanteau theorem.

3. Exercise 9.1.6, PfS Course Notes, page 197. (Exercise 11.7.6, page 291, PfS 2000).

   Solution: Theorem 9.1.5 as stated in PfS (Course Notes) seems to have a (minor) wobble. Here is a restatement:
   Theorem 1.5. (Convergence of types). Suppose that $(X_n - b_n)/a_n \to_d X \sim F$ and $(X_n - \beta_n)/\alpha_n \to Y \sim G$, where $a_n > 0$,

$\alpha_n > 0$, and both X and Y are nondegenerate. Then there exists $a > 0$ and a real b such that

$$a_n/\alpha_n \to (\text{some positive } a) \qquad \text{and} \qquad (\beta_n - b_n)/a_n \to (\text{some real } b) \quad (1)$$

and $Y \stackrel{d}{=} aX + b$ (or, equivalently, $G(y) = F((y-b)/a)$ for all y).

Proof. Suppose first that $Z_n \equiv (X_n - b_n)/a_n \to X \sim F$ and that (1) holds. Then

$$\begin{aligned} \widetilde{Z}_n \equiv \frac{X_n - \beta_n}{\alpha_n} &= \frac{a_n}{\alpha_n}\frac{X_n - b_n}{a_n} + \frac{b_n - \beta_n}{a_n} \\ &\to_d \; aX + b = Y \end{aligned}$$

so $(Y-b)/a \stackrel{d}{=} X \sim F$ and

$$G(y) = P(Y \le y) = P(aX+b \le y) = P(X \le (y-b)/a) = F((y-b)/a).$$

On the other hand, suppose that $Z_n \equiv (X_n - b_n)/a_n \to_d X \sim F$ and $(X_n - \beta_n)/\alpha_n \to Y \sim G$ with both F and G non-degenerate. Let $F_n(z) \equiv P(Z_n \le z)$. Then, since

$$\begin{aligned} \widetilde{Z}_n \equiv \frac{X_n - \beta_n}{\alpha_n} &= \frac{a_n}{\alpha_n}\frac{X_n - b_n}{a_n} + \frac{b_n - \beta_n}{a_n} \\ &\equiv c_n Z_n + d_n \to_d Y \end{aligned}$$

by hypothesis, it follows that

$$0 < \inf_n \frac{a_n}{\alpha_n} \le \sup_n \frac{a_n}{\alpha_n} < \infty, \qquad \text{and} \qquad \sup_n |\frac{\beta_n - b_n}{a_n}| < \infty;$$

that is

$$0 < \inf_n c_n \le \sup_n c_n < \infty, \qquad \text{and} \qquad \sup_n d_n < \infty. \qquad (2)$$

To see this, suppose that $\{c_n\}$ contains a subsequence $\{c_{n'}\}$ such that $c_{n'} \to 0$. Then $\widetilde{Z}_{n'} = o_p(1) + d_{n'} \to \pm\infty$ if $d_{n'}| \to \pm\infty$, violating the hypothesis that $\widetilde{Z}_n \to_d Y \sim G$ non-degenerate. If $d_{n'}$ is bounded, then by extracting a further subsequence $\{d_{n''}\}$ such that $d_{n''} \to$ some d_0, a constant, and hence then $\widetilde{Z}_{n''} = o_p(1) + d_{n''} \to_d d_0$, which also violates

$\widetilde{Z}_n \to_d Y$ non-degenerate. Thus $\{c_n\}$ is bounded away from 0. By reversing the argument we can write

$$Z_n = c_n^{-1} \widetilde{Z}_n - d_n$$

and conclude that $\{c_n^{-1}\}$ is bounded away from 0, and hence $\{c_n\}$ is bounded away from both 0 and ∞. If $\{d_n\}$ is not bounded, choose a subsequence $\{d_{n'}\}$ such that $d_{n'} \to \pm\infty$ and $a_{n'} \to$ some a. But then $\widetilde{Z}_n \to_{sd} a \cdot X \pm \infty$ which contradicts the hypothesis $\widetilde{Z}_n \to_d Y$ nondegenerate. Thus (2) holds, and this is equivalent to (2).

Now fix a subsequence $\{n'\}$ along which $a_{n'}/\alpha_{n'} \to a > 0$ and $(\beta_{n'} - b_{n'})/a_{n'} \to b$. Then along this subsequence

$$\widetilde{Z}_n = \frac{X_{n'} - \beta_{n'}}{\alpha_{n'}} \to_d aX + b \equiv Y$$

and hence $G(y) = F((y-b)/a)$ for all y or $F(x) = G(ax+b)$. Suppose that along some other subsequence $\{n''\}$ we have

$$a_{n''}/\alpha_{n''} \to a' > 0 \qquad \text{and} \qquad (\beta_{n''} - b_{n''})/a_{n''} \to b'.$$

Then by reasoning as before we conclude that also $Y \stackrel{d}{=} a'X + b'$. Combining this with $Y \stackrel{d}{=} aX + b$ yields $(a' - a)X + (b' - b) \stackrel{d}{=} 0$, and since X is nondegenerate this implies $a' = a$ and $b' = b$. Thus every convergent subsequence of $\{(a_n/\alpha_n, (\beta_n - b_n)/a_n)\}$ converges to (a, b), and so the whole sequence converges to (a, b). Thus $Y = aX + b)$ or $F(x) = G(ax + b)$ for all x. $\square$

4. Suppose that $X_1, \ldots, X_m$ are i.i.d. with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$; suppose that $Y_1, \ldots, Y_n$ are i.i.d. and independent of the X_i's with $E(Y_1) = \nu$ and $Var(Y_1) = \tau^2 < \infty$.
(a) Use the classical CLT (Theorem 11.2.2, W Chapter 11) to show that $\sqrt{m}(\overline{X}_m - \mu)/\sigma \to_d N(0,1)$ and that $\sqrt{n}(\overline{Y}_n - \nu)/\tau \to_d N(0,1)$.
(b) Let $N = m + n$ and set

$$D_{m,n} \equiv \sqrt{\frac{mn}{N}} \left(\overline{Y}_n - \overline{X}_m - (\nu - \mu) \right).$$

Use (a) to show that if $\lambda_N \equiv m/N \to \lambda \in [0,1]$ then

$$D_{m,n} \to_d \sqrt{\lambda}\tau Z - \sqrt{1-\lambda}\sigma Z' \sim N(0, \lambda\tau^2 + (1-\lambda)\sigma^2)$$

where $Z, Z' \sim N(0,1)$ are independent.
(c) Let $S^2_{m,n} \equiv \lambda_N S^2_Y + (1-\lambda_N) S^2_X$ where $S^2_X \equiv m^{-1}\sum_{i=1}^m (X_i - \overline{X}_m)^2$ and $S^2_Y \equiv n^{-1}\sum_{j=1}^n (Y_j - \overline{Y}_n)^2$. Show that if $\lambda_N \to \lambda \in [0,1]$ then $S^2_{m,n} \to_p \lambda\tau^2 + (1-\lambda)\sigma^2$.
(d) Use (b) and (c) to show that if $\lambda_N \to \lambda \in [0,1]$, then $T_{m,n} \equiv D_{m,n}/S_{m,n} \to_d Z \sim N(0,1)$.
(e) Use the result of (d) and the Helly selection theorem to show that $T_{m,n} \to_d Z \sim N(0,1)$ whenever $m \to \infty$ and $n \to \infty$.

Solution: (a) The classical CLT yields the claimed convergences:

$$\sqrt{m}(\overline{X}_m - \mu)/\sigma \to_d N(0,1), \qquad \text{and} \qquad \sqrt{n}(\overline{Y}_n - \nu)/\tau \to_d N(0,1).$$

By independence of the X's and Y's we also have joint convergence, and this can be written as

$$\begin{pmatrix} \sqrt{m}(\overline{X}_m - \mu) \\ \sqrt{n}(\overline{Y}_n - \nu) \end{pmatrix} \to_d \begin{pmatrix} \sigma Z \\ \tau Z' \end{pmatrix}$$

where Z, Z' are independent $N(0,1)$ rv's.
(b) We can write

$$\begin{aligned} D_{m,n} &\equiv \sqrt{\frac{mn}{N}}\left(\overline{Y}_n - \overline{X}_m - (\nu - \mu)\right) \\ &= \sqrt{m/N}\sqrt{n}(\overline{Y}_n - \nu) - \sqrt{n/N}\sqrt{m}(\overline{X}_m - \mu) \\ &\to_d \sqrt{\lambda}\tau Z' - \sqrt{1-\lambda}\sigma Z \\ &\sim N(0, \lambda\tau + (1-\lambda)\sigma) \end{aligned}$$

if $\lambda_N \equiv m/N \to \lambda$ by the continuous mapping theorem and the convergence in (a).
(c) Now $S^2_X \to_p \sigma^2$ and $S^2_Y \to_p \tau^2$ by the weak law of large numbers (applied to $(X_i - \mu)^2$ and $(Y_i - \nu)^2$ respectively) and the continuous mapping for convergence in probability. Therefore if $\lambda_N \to \lambda \in [0,1]$ we have

$$S^2_{m,n} \equiv \lambda_N S^2_Y + (1-\lambda_N) S^2_X \to_p \lambda\sigma^2 + (1-\lambda)\tau^2$$

by continuous mapping for convergence in probability.
(d) Now (b) and (c) yield

$$T_{m,n} \equiv \frac{D_{m,n}}{S_{m,n}} \to_d \frac{\sqrt{\lambda}\tau Z' - \sqrt{1-\lambda}\sigma Z}{\sqrt{\lambda\tau^2 + (1-\lambda)\sigma^2}} \sim N(0,1)$$

by Slutsky's theorem, assuming that $\lambda_N \to \lambda \in [0,1]$.
(e) If $m \to \infty$ and $n \to \infty$ in any way, it remains true that $\lambda_N = m/N \in [0,1]$ for all m, n. Thus for any subsequences $\{m'\}$ and $\{n'\}$ there exist further subsequences $\{m''\}$ and $\{n''\}$such that $\lambda_{N''} \equiv m''/N'' \to$ some $\lambda \in [0,1]$. For these subsequences we conclude from (a)-(d) that $T_{m'',n''} \to_d N(0,1)$. Since the limit is the same for all such initial subsequences $\{m'\}$ and $\{n'\}$, we conclude that the full sequence $T_{m,n}$ satisfies $T_{m,n} \to_d N(0,1)$.