Statistics 522, Problem Set 6 Solutions

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1. PfS Course notes, Exercise 9.1.1 page 195. (PfS 2000, Exercise 11.7.1, page 289).

   (a) Suppose $F_n \to_{sd} H$ and that both $F_{n-}(a) \to H_-(a)$ and $F_n(b) \to H(b)$ for some constants $-\infty < a < b < \infty$ in C_H having $H(a) < H(b)$. Then

   $$\int_{[a,b]} g dF_n \to \int_{[a,b]} g dH$$

   for all $g \in C_{[a,b]} \equiv \{g: \ g \text{is continuous on } [a,b]\}$. Moreover, if $F_n \to_{sd} H$, then

   $$\int g dF_n \to \int g dH \qquad \text{for all} \ \ g \in C_0$$

   where $C_0 \equiv \{g: \ g \text{ is continuous on } \ R \text{ and } g(x) \to 0 \text{ as } |x| \to \infty\}$.

   (b) Suppose $F_n \to_d F$ and g is continuous on the line. Suppose $|g(x)|/\psi(x) \to 0$ as $|x| \to \infty$, where $\psi \ge 0$ has $\int \psi dF_n \le K$ for all n. Then $\int g dF_n \to \int g dF$.

   (c) If $E|X_n|^{r_0} < (\text{some} M) < \infty$ for all large n, then $F_n \to_d F$ implies that

   $$E|X_n|^r \to E|X|^r \quad \text{and} \quad EX_n^k \to EX^k \ \text{ for } 0 < r < r_0 \ \text{ and } \ 0 < k < r_0.$$

   (d) Let g be continuous. If $F_n \to_{sd} H$, then $\int |g| dH \le \liminf \int |g| dF_n$. Actually, g continuous a.s. H suffices in (a), (b), and (d) above.

   Solution: (a) Fix $\delta > 0$ so that $a - \delta \in C_H$ and $b + \delta \in C_H$. Then define

   $$\widetilde{F}_n(x) = \begin{cases} 1, & x \ge b + \delta \\ F_n(b), & x \in [b, b+\delta) \\ F_n(x), & x \in [a,b] \\ F_n(a), & x \in [a-\delta, a) \\ 0, & x < a - \delta. \end{cases}$$

Define $\widetilde{F}$ analogously in terms of H. Then $\widetilde{F}$ is a distribution function and the hypotheses imply that $\widetilde{F}_n \to_d \widetilde{F}$. Now define $\widetilde{g}$ by $\widetilde{g}(x) = g(x)1_{[a,b]}(x)$. Then $\widetilde{g}$ is continuous a.s. $\widetilde{F}$ and it follows from the Helly-Bray theorem that

$$\begin{aligned}\int_{[a,b]} g dF_n &= \int_{[a,b]} \widetilde{g} d\widetilde{F}_n \\ &= \int \widetilde{g} d\widetilde{F}_n \to \int \widetilde{g} d\widetilde{F} = \int_{[a,b]} g dH.\end{aligned}$$

Now suppose $g \in C_0$. Since $g(x) \to 0$ as $|x| \to \infty$, we can choose $a \in C_H$ and $b \in C_H$ so that $|g(x)| \le \epsilon/8$ for all $x \ge a$ and all $x \ge b$. Then write

$$\int g dF_n = \int_{(-\infty,a)} g dF_n + \int_{(b,\infty)} g dF_n + \int_{[a,b]} g dF_n,$$

$$\int g dH = \int_{(-\infty,a)} g dH + \int_{(b,\infty)} g dH + \int_{[a,b]} g dH.$$

By the argument above we can choose n so large that

$$\Big| \int_{[a,b]} g dF_n - \int_{[a,b]} g dH \Big| \le \frac{\epsilon}{2}$$

for all $n \ge$ some N_ϵ. Then, for $n \ge N_\epsilon$,

$$\begin{aligned}&\Big| \int g dF_n - \int g dH \Big| \\ &\le \Big| \int_{[a,b]} g dF_n - \int_{[a,b]} g dH \Big| + \Big| \int_{(-\infty,a)} g dF_n \Big| + \Big| \int_{(b,\infty)} g dF_n \Big| \\ &\qquad + \Big| \int_{(-\infty,a)} g dH \Big| + \Big| \int_{(b,\infty)} g dH \Big| \\ &\le \epsilon/2 + (F_n(a) + 1 - F_n(b) + H(a) + 1 - H(b))\epsilon/8 \\ &\le \epsilon/2 + 4(\epsilon/8) = \epsilon,\end{aligned}$$

and hence the claimed convergence holds.
(b) Fix $\epsilon > 0$ and $\delta = \epsilon/(4K)$. Note that

$$\begin{aligned}\int \psi dF &= E\psi(X) = E \liminf_n \psi(X_n) \le \liminf_n E\psi(X_n) \\ &\le \limsup_n E\psi(X_n) = \limsup_n \int \psi dF_n \le K \qquad (1)\end{aligned}$$

since $\int \psi dF_n \leq K$ for all n. Here we assumed (without loss of generality) that ψ is continuous and used a Skorokhod construction argument together with Fatou's lemma. Now choose $\pm M \in C_F$ so large that $|g(x)| \leq \delta\psi(x)$ for all $x \in [-M, M]^c$. Then we have

$$\begin{aligned} \Big|\int g dF_n - \int g dF\Big| &\leq \Big|\int_{[-M,M]} g dF_n - \int_{[-M,M]} g dF\Big| \\ &\quad + \Big|\int_{[-M,M]^c} g dF_n\Big| + \Big|\int_{[-M,M]^c} g dF\Big| \\ &\leq \Big|\int_{[-M,M]} g dF_n - \int_{[-M,M]} g dF\Big| + 2\delta K \\ &\leq \Big|\int_{[-M,M]} g dF_n - \int_{[-M,M]} g dF\Big| + \epsilon/2 \qquad (2) \end{aligned}$$

by our choice of δ and

$$\begin{aligned} \Big|\int_{[-M,M]^c} g dF_n\Big| &\leq \int_{[-M,M]^c} \frac{|g|}{\psi} \psi dF_n \leq \delta \int_{[-M,M]^c} \psi dF_n \\ &\leq \delta \int \psi dF_n \leq \delta K \end{aligned}$$

and similarly $|\int_{[-M,M]^c} g dF| \leq \delta K$ by using (1). But the first term in (2) converges to 0 as $n \to \infty$ by (a) above, and hence is less than $\epsilon/2$ for $n \geq$ some N_ϵ. Hence the claimed convergence holds.

(c) Take $g(x) = |x|^r$ and $\psi(x) = |x|^{r_0}$ with $0 < r < r_0$. Then

$$\frac{|g(x)|}{\psi(x)} = |x|^{r - r_0} \to 0 \quad \text{as} \quad |x| \to \infty.$$

Similarly, if $g(x) = x^k$ and $\psi(x) = |x|^{r_0}$ with $0 < k < r_0$. Then

$$\frac{|g(x)|}{\psi(x)} = |x|^{k - r_0} \to 0 \quad \text{as} \quad |x| \to \infty.$$

Thus the hypotheses of (b) above hold with these choices of g and ψ. Thus it follows that $E|X_n|^r \to E|X|^r$ and $EX_n^k \to EX^k$.

(d) This is analogous to the similar fact for the case of $F_n \to_d F$, namely

$$\int |g| dF \leq \liminf_n \int g dF_n,$$

which is easily proved via a Skorokhod construction. In the current setting, $F_n \to_{sd} H$, then there exist random variables $X_n^* \overset{d}{=} X_n \sim F_n$ and a possibly extended valued random variable $X^* \overset{sd}{=} X \sim H$, all defined on a common probability space, such that $X_n^* \to_{a.s.} X^*$. Then

$$\int |g| dH = E|g(X)| \quad = \quad E \liminf |g(X_n^*)| \leq \liminf E|g(X_n^*)| = \liminf \int |g| dF_n.$$

2. PfS Course notes, Exercise 9.1.2 page 195. (PfS 2000, Exercise 11.7.2, page 289). (Pólya's lemma) If $F_n \to_d F$ for a continuous df F, then

$$\|F_n - F\| = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0.$$

Thus if $F_n \to_d F$ with F continuous and $x_n \to x$, then $F_n(x_n) \to F(x)$.

Solution: Let M be a (large) positive integer, and set $x_{M,j} \equiv x_j = F^{-1}(j/(M+1))$ for $j = 1, \ldots, M$, and let $x_{M,0} \equiv -\infty$, $x_{M,M+1} \equiv \infty$. Then for $x \in [x_{j-1}, x_j]$ we have

$$F_n(x) - F(x) \begin{cases} \leq F_n(x_j) - F(x_{j-1}) \leq F_n(x_j) - F(x_j) + 1/M \\ \geq F_n(x_{j-1}) - F(x_j) \geq F_n(x_{j-1}) - F(x_{j-1}) - 1/M \end{cases},$$

and hence

$$\begin{aligned} \|F_n - F\| &\leq \max_{1 \leq j \leq M+1} \sup_{x_{j-1} \leq x \leq x_j} |F_n(x) - F(x)| \\ &\leq \max_{1 \leq j \leq M} |F_n(x_j) - F(x_j)| + 1/M \\ &\to 0 + 1/M \end{aligned}$$

since $F_n(x) \to F(x)$ at all x in view of F being continuous. But since M is arbitrary, this can be made arbitrarily small; i.e. $\|F_n - F\| \to 0$.

3. PfS Course notes, Exercise 9.1.3 page 195. (PfS 2000, Exercise 11.7.3, page 289). Suppose $X_n \sim F_n$. Show that $\{F_n : \ n \geq 1\}$ is tight if either: (a) $\limsup_n E|X_n|^r < \infty$ for some $r > 0$; or (b) $F_n \to_d F$.

Solution: (a) Let $\epsilon > 0$. Now by Markov's inequality

$$\limsup_{n\to\infty} P(|X_n| > M) \leq \frac{\limsup_{n\to\infty} E|X_n|^r}{M^r} < \epsilon$$

for $M > M(r, \epsilon) \equiv (\limsup_{n\to\infty} E|X_n|^r/\epsilon)^{1/r}$. Thus there is an $N \equiv N(\epsilon, r)$ such that $\sup_{n>N} P(|X_n| > M) \leq 2\epsilon$. But since $F_1, \ldots, F_N$ are df's, there exists a $K = K_\epsilon$ so large that $\max_{1\leq n\leq N} P(|X_n| > K) < 2\epsilon$. Taking $R = R_\epsilon = \max\{M, K\}$ we have

$$\sup_{1\leq n<\infty} P(|X_n| > R) < 2\epsilon\,.$$

Thus $\{F_n\}$, the family of distributions of $\{X_n\}$, is tight.
(b) Let $\epsilon > 0$. Let $r = r(\epsilon) \in C_F$ be so large that $1 - F(r) < \epsilon/4$, and let $l = l(\epsilon) \in C_F$ be so small that $F(l) < \epsilon/4$. Now there exists an $N = N_\epsilon$ so large that

$$|F_n(r) - F(r)| < \epsilon/4 \qquad \text{for all} \qquad n > N$$

and

$$|F_n(l) - F(l)| < \epsilon/4 \qquad \text{for all} \qquad n > N\,.$$

Then we have

$$\begin{aligned}\sup_{n>N} F_n([l, r]^c) &\leq F(l) + (1 - F(r)) \\ &\qquad + |F_n(l) - F(l)| + |F_n(r) - F(r)| \\ &\leq \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon\end{aligned}$$

But since $F_1, \ldots, F_N$ are distribution functions, we can easily find an interval $[l', r']$, such that

$$\max_{1\leq n\leq N} F_n([l', r']^c) < \epsilon\,,$$

and hence the interval $[a, b] \equiv [l \wedge l', r \vee r']$ satisfies

$$\sup_{1\leq n<\infty} F_n([a, b]^c) < \epsilon\,;$$

i.e $\{F_n\}$ is tight.

4. Exercise 11.6.1, page 34, Wellner, Chapter 11, notes.
Show that (i) $F_n(x) = P(X_n \le x) \to P(X \le x) = F(x)$ for all $x \in C_F$ if and only if (ii) $Ef(X_n) \to Ef(X)$ for all $f \in C_b(\mathbb{R})$.

Solution: Suppose that (ii) holds. Let

$$h(x) = \begin{cases} 1, & x < 0, \\ 1 - x, & 0 \le x \le 1, \\ 0, & x > 1, \end{cases}$$

and for fixed $x \in \mathbb{R}$ and $\epsilon > 0$, consider the functions

$$f_{\epsilon,x}(y) = h\left(\frac{y-x}{\epsilon}\right),$$
$$g_{\epsilon,x}(y) = h\left(\frac{y-x+\epsilon}{\epsilon}\right).$$

Note that $f_{\epsilon,x}$ and $g_{\epsilon,x}$ are continuous and bounded (since their range is $[0,1]$. Moreover

$$1_{(-\infty,x-\epsilon]}(y) \le g_{\epsilon,x}(y) \le 1_{(-\infty,x]}(y) \le f_{\epsilon,x}(y) \le 1_{(-\infty,x+\epsilon]}(y) \qquad \text{for all } y \in \mathbb{R}.$$

Thus by (ii) $Ef_{\epsilon,x}(X_n) \to Ef_{\epsilon,x}(X)$ and $Eg_{\epsilon,x}(X_n) \to Eg_{\epsilon,x}(X)$. Therefore

$$F_n(x) = E1_{(-\infty,x]}(X_n) \le Ef_{\epsilon,x}(X_n) \to Ef_{\epsilon,x}(X) \le E1_{(-\infty,x+\epsilon]}(X) = F(x+\epsilon),$$

and, similarly,

$$F(x-\epsilon) \le Eg_{\epsilon,x}(X) \leftarrow Eg_{\epsilon,x}(X_n) \le E1_{(-\infty,x]}(X_n) = F_n(x).$$

Therefore we conclude that

$$F(x-\epsilon) \le \liminf F_n(x) \le \limsup F_n(x) \le F(x+\epsilon)$$

for every $\epsilon > 0$. Letting $\epsilon \to 0$ and assuming that x is a continuity point of F, we conclude that $F_n(x) \to F(x)$; i.e. (i) holds.

For the reverse implication (i) implies (ii) (which is also solved via the Skorokhod theorem on PfS, page 53), choose $I = [a,b]$ with $a, b \in C_F$

such that $P_F(I^c) = F(a) + (1 - F(b)) \leq \epsilon$. Let $a = a_0 < a_1 \cdots < a_m = b$ define a partition of $\{I_j = (a_{j-1}, a_j] : 1 \leq j \leq m\}$ of $[a, b]$ with $a_j \in C_F$ for all $0 \leq j \leq m$. Based on this partition define approximating functions f_m^+ and f_m^- by

$$\begin{aligned} f_m^+(x) &\equiv \sum_{j=1}^m \sup_{x \in I_j}(f(x)) 1_{I_j}(x), \\ f_m^-(x) &\equiv \sum_{j=1}^m \inf_{x \in I_j}(f(x)) 1_{I_j}(x). \end{aligned}$$

Then

$$\int_I f_m^- dF_n \leq \int_I f dF_n \leq \int_I f_m^+ dF_n.$$

The left and right sides in the last display converge, in view of our hypothesis, and hence

$$\int_I f_m^- dF \leq \liminf_{n\to\infty} \int_I f dF_n \leq \limsup_{n\to\infty} \int_I f dF_n \leq \int_I f_m^+ dF.$$

Now let $\max_{j\leq m} |a_j - a_{j-1}| \to 0$. Then, by choosing the partitions to be nested,

$$f_m^-(x) \uparrow f(x), \qquad \text{and} \qquad f_m^+(x) \downarrow f(x).$$

By the dominated convergence this yields

$$\lim_m \int_I f_m^+ dF = \int_I f dF \qquad \text{and} \qquad \lim_m \int_I f_m^- dF = \int_I f dF.$$

Thus we conclude that

$$\begin{aligned} \limsup_{n\to\infty} \Big| \int f dF_n - \int f dF \Big| &\leq \limsup_{n\to\infty} \Big| \int_I f dF_n - \int_I f dF \Big| + \|f\|_\infty \cdot 2\epsilon \\ &= 2\epsilon \|f\|_\infty. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this yields $\int f dF_n \to \int f dF$ for all bounded and continuous F.

5. Exercise 11.6.2, page 34, Wellner, Chapter 11, notes: Suppose that $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$ where both μ and σ^2 are finite. Suppose that $Z \sim P_0$ on $\mathbb{R}$.
   (a) Show that $X_n \overset{d}{=} \mu_n + \sigma_n Z \to_d \mu + \sigma Z \overset{d}{=} X$.
   (b) Show that for $f \in BL(\mathbb{R})$

$$|Ef(X_n) - Ef(X)| \le \|f\|_{BL} E\{1 \wedge (|\mu_n - \mu| + |\sigma_n - \sigma||Z|)\}.$$

Solution: (b) For $f \in BL(\mathbb{R})$ it follows that $|f(y) - f(x)| \le \|f\|_{BL}\{1 \wedge |y - x|\}$ for all $x, y \in \mathbb{R}$; recall the inequality before Definition 1.4 on page 4. Thus

$$|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \le \|f\|_{BL}\{1 \wedge |\mu_n - \mu + (\sigma_n - \sigma)Z|\};$$

This implies that

$$\begin{aligned} |Ef(X_n) - Ef(X)| &= |Ef(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\ &\le E|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\ &\le \|f\|_{BL} E\{1 \wedge |\mu_n - \mu| + |\sigma_n - \sigma||Z|\}. \end{aligned}$$

(a) Since $\mu_n - \mu \to 0$ and $\sigma_n - \sigma \to 0$, it follows by the dominated convergence theorem (with dominating function 1) that the right side of the last display converges to 0. Thus $Ef(X_n) \to Ef(X)$ for all $f \in BL(\mathbb{R})$; therefore $X_n \to_d X$ by the portmanteau theorem.