Statistics 522, Problem Set 5 Solutions

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1. Polyá's urn: At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn, and is replaced together with a new ball of the same color. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n + 1)/(n+2)$, the proportion of black balls in the urn just after time n. Prove that (relative to a natural filtration which you should specify) M_n is a martingale. Prove that $P(B_n = k) = 1/(n+1)$ for $0 \le k \le n$. What is the distribution of $\Theta \equiv \lim_n M_n$? Prove that for $0 < \theta < 1$,

$$N_n^\theta \equiv \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}$$

defines a martingale N_n^θ.

Solution: Let $\mathcal{F}_n \equiv \sigma(B_1, \ldots, B_n)$. Note that $M_n \equiv (B_n+1)/(n+2)$ is the conditional (given $\mathcal{F}_n$) probability of drawing a black ball at the $n+1$st draw. Thus we compute

$$\begin{aligned}
E(M_{n+1}|\mathcal{F}_n) &= E\left(\frac{B_{n+1}+1}{n+3}|\mathcal{F}_n\right) = \frac{1}{n+3}E(B_{n+1}+1|\mathcal{F}_n) \\
&= \frac{1}{n+3}\{(B_n+1)(1-M_n) + (B_n+2)M_n\} \\
&= \frac{1}{n+3}\{B_n + 1 - M_n + 2M_n\} \\
&= \frac{1}{n+3}\{(n+2)M_n + M_n\} = M_n \qquad \text{a.s.}.
\end{aligned}$$

Hence $\{M_n, \mathcal{F}_n\}$ is a martingale. Similarly, letting

$$p_n(k) \equiv \frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k},$$

the process $N_n^\theta = p_n(B_n)$ and

$$
\begin{aligned}
E(N_{n+1}^\theta|\mathcal{F}_n) &= E(p_{n+1}(B_{n+1})|\mathcal{F}_n) \\
&= p_{n+1}(B_n)(1-M_n) + p_{n+1}(B_n+1)M_n \\
&= \frac{(n+2)!}{B_n!(n+1-B_n)!}\theta^{B_n}(1-\theta)^{n+1-B_n}\frac{(n+1-B_n)}{(n+2)} \\
&\qquad + \frac{(n+2)!}{(B_n+1)!(n+1-B_n-1)!}\theta^{B_n+1}(1-\theta)^{n+1-B_n-1}\frac{(B_n+1)}{(n+2)} \\
&= \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\{(1-\theta)+\theta\} \\
&= p_n(B_n) \equiv N_n^\theta \qquad \text{a.s.},
\end{aligned}
$$

so $\{N_n^\theta, \mathcal{F}_n\}$ is a martingale. This implies that $EN_n^\theta = EN_0^\theta = 1$ for all $\theta \in (0,1)$, or

$$
E\left\{\frac{n!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}\right\} = \frac{1}{n+1}\,. \tag{1}
$$

This equality clearly holds if $P(B_n = k) = 1/(n+1)$ for $k = 0, \ldots, n$. On the other hand, (1.1) implies, by letting $\alpha = \theta/(1-\theta)$, that, with $p_k = P(B_n = k)$,

$$
\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\alpha^k p_k = \frac{1}{n+1}(1+\alpha)^n = \frac{1}{n+1}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\alpha^k\,,
$$

and this yields $p_k = 1/(n+1)$ by matching coefficients.

The distribution of B_n is a discrete uniform distribution on $0, \ldots, n$ for every n, so the distribution of M_n is a discrete uniform distribution on $0 < 1/(n+1) < \cdots < (n+1)/(n+2) < 1$ and it is clear that $M_n \to_d U(0,1)$ as $n \to \infty$; $P(M_n \le u) = [(n+2)u]/(n+1) \to u = P(U \le u)$ where $U \sim \text{Uniform}(0,1)$.

2. Let $X_1, X_2, \ldots$ be i.i.d rv's with $P(X=1) = p$, $P(X=-1) = 1-p \equiv q$, where $0 < p < 1$ and $p \neq q$. Suppose that a, b are integers with $-a < 0 < b$. Define

$$
S_n = X_1 + \ldots + X_n, \qquad T \equiv \inf\{n : S_n = -a, \text{or} \;\; S_n = b\}\,.
$$

Let $\mathcal{F}_n \equiv \sigma[X_1, \ldots, X_n]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prove that $M_n \equiv (q/p)^{S_n}$ and $N_n = S_n - n(p-q)$ define martingales M_n and N_n. How would you use these martingales to deduce the values of $P(S_T = -a)$ and $E(S_T)$? [Hint: see PfS, Course Notes, pages 381-382.]

Solution: T is clearly a stopping time and, for each n

$$P(T \le n + b | \mathcal{F}_n) \ge p^{b - S_n} + q^{S_n} \ge (p \wedge q)^b \equiv \epsilon > 0$$

since $p \in (0,1)$.

Thus the hypotheses of Williams PwM, E10.5, page holds with $N = b$ and $\epsilon \equiv (p \wedge q)^b$. Thus $E(T) < \infty$, and the third set of sufficient conditons for Doob's optional sampling theorem hold. Since $\{S_n - n(p-q), \mathcal{F}_n\}$ and $\{(q/p)^{S_n}, \mathcal{F}_n\}$ are both martingales, we conclude from Doob's optional sampling theorem that

$$E\left(\frac{q}{p}\right)^{S_T} = E\left(\frac{q}{p}\right)^{S_0} = \left(\frac{q}{p}\right)^0 = 1\,. \qquad (2)$$

But the left side of (2) equals

$$\left(\frac{q}{p}\right)^b P(S_T = b) + \left(\frac{q}{p}\right)^{-a} P(S_T = -a) \equiv \left(\frac{q}{p}\right)^b p_b + \left(\frac{q}{p}\right)^{-a} (1 - p_b)\,.$$

Thus we can solve for p_b to obtain

$$p_b = P(S_T = b) = \frac{1 - (q/p)^{-a}}{(q/p)^b - (q/p)^{-a}} = \frac{(q/p)^a - 1}{(q/p)^{a+b} - 1}\,,$$

and

$$p_a = P(S_T = -a) = 1 - p_b = \frac{(q/p)^b - 1}{(q/p)^b - (q/p)^{-a}} = \frac{1 - (p/q)^b}{1 - (p/q)^{a+b}}\,.$$

It follows that

$$\begin{aligned} E(S_T) &= bp_b + (-a)p_a \\ &= b\left(1 - \frac{1 - (p/q)^b}{1 - (p/q)^{a+b}}\right) - a\frac{1 - (p/q)^b}{1 - (p/q)^{a+b}} \\ &= b - (a+b)\frac{1 - (p/q)^b}{1 - (p/q)^{a+b}}\,. \end{aligned}$$

Since $\{S_n - n(p-q)\} = \{S_n - n\mu\}$ is a martingale, we deduce that $E(S_T - T\mu) = 0$ and hence that

$$E(T) = \frac{1}{\mu} E(S_T) = \frac{1}{\mu}\left\{ b - (a+b)\frac{1-(p/q)^b}{1-(p/q)^{a+b}} \right\} .$$

You should also take a look at the situation for $p = q = 1/2$ in Section 13.7, page 381.

3. Exercise 13.7.2, PfS, Course Notes page 382. Suppose that S_μ is Brownian motion with drift: $S_\mu(t) = S(t) + \mu t$ for $t \geq 0$. Let $\tau_{ab} \equiv \tau \equiv \inf\{t \geq 0 : \; S_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$.
Claim 1: $S_0(t)$, $S_0^2(t) - t$, $S_\mu(t) - \mu t$ are mean 0 martingales, and, with $\theta = -2\mu$,

$$\exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$$

is a mean 1 martingale.
Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a+b)$ and $E\tau = ab$.
Claim 3: If $\mu \neq 0$, then

$$P(S(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$

and

$$E(\tau) = \frac{b}{\mu} - \frac{a+b}{\mu}\frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}} .$$

Claim 4: If $\mu < 0$, then $P(\|S_\mu^+\|_0^\infty \geq b) = P(\sup_{0 \leq t < \infty} S_\mu(t) \geq b) = \exp(-2|\mu| b)$ for all $b > 0$; i.e. $\|S_\mu\|_0^\infty \sim \text{Exponential}(2|\mu|)$.
(Note the analogies with problem 2.)

Solution: Proof of claim 1: Since standard Brownian motion S has independent increments, with $\mathcal{A}_t \equiv \sigma[S(s), 0 \leq s \leq t]$ we have, for $0 \leq s \leq t$,

$$\begin{aligned} E(S(t)|\mathcal{A}_s) &= E(S(t) - S(s) + S(s)|\mathcal{A}_s) \\ &= E(S(t) - S(s)|\mathcal{A}_s) + E(S(s)|\mathcal{A}_s) \\ &= E(S(t) - S(s)) + S(s) = 0 + S(s) = S(s) \qquad \text{a.s.} \end{aligned}$$

so that $\{S(t), \mathcal{A}_t\}_{t\geq 0}^{\infty}$ is a zero - mean martingale. Since $S_\mu(t) - \mu t = S_0(t) = S(t)$, it follows immediately that $\{S_\mu(t) - \mu t, \mathcal{A}_t\}_{t\geq 0}^{\infty}$ is also a 0-mean martingale. To see that $\{S^2(t) - t, \mathcal{A}_t\}_{t\geq 0}^{\infty}$ is a zero-mean martingale, we calculate

$$\begin{aligned}
E(S^2(t) - t|\mathcal{A}_s) &= E([S(t) - S(s) + S(s)]^2 - (t - s + s)|\mathcal{A}_s) \\
&= E((S(t) - S(s))^2 - (t - s)|\mathcal{A}_s) \\
&\qquad + E(2(S(t) - S(s))S(s)|\mathcal{A}_s) \\
&\qquad + E(S^2(s) - s|\mathcal{A}_s) \\
&= E(S(t) - S(s)^2) - (t - s) + 2S(s)E(S(t) - S(s)) \\
&\qquad + (S^2(s) - s) \\
&= 0 + 0 + S^2(s) - s = S^2(s) - s \qquad \text{a.s.}
\end{aligned}$$

so that the claim holds. (Note that this shows that $\langle S\rangle(t) = t$ is the predictable variation process corresponding to the sub - martingale $S^2(t)$.) To see that $Y_t = \exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$ is a mean 1 martingale, note that $Y_t = \exp(\theta S(t) - \theta^2 t/2)$ and hence

$$\begin{aligned}
E(Y_t|\mathcal{A}_s) &= E(\exp(\theta(S(t) - S(s)))|\mathcal{A}_s) \cdot E(\exp(\theta S(s) - \theta^2 s/2)|\mathcal{A}_s) \\
&\qquad \cdot \exp(\theta^2(s/2 - t/2)) \\
&= E(\exp(\theta(S(t) - S(s)))) \cdot \exp(\theta^2(s/2 - t/2)) \cdot Y_s \qquad \text{a.s.} \\
&= \exp(\theta^2(t/2 - s/2)) \cdot \exp(\theta^2(s/2 - t/2)) \cdot Y_s \qquad \text{a.s.} \\
&= Y_s\,,
\end{aligned}$$

so that Y_t is a mean 1 mg. The second part of this holds simply because, with $\theta = -2\mu$ we have

$$\theta[S_\mu - \mu t] - \theta^2 t/2 = -2\mu S_\mu + 2\mu^2 t - 4\mu^2 t/2 = -2\mu S_\mu(t)\,.$$

Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a + b)$ and $E\tau = ab$.
Claim 3: If $\mu \neq 0$, then

$$P(S(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$

and

$$E(\tau) = \frac{b}{\mu} - \frac{a + b}{\mu}\frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}\,.$$

To prove claim 2, first consider the bounded stopping times $\tau \wedge k$. Then by the basic optional sampling theorem,

$$0 = E(S_0^2(\tau \wedge k) - \tau \wedge k)\,. \tag{3}$$

Now $\tau \wedge k \nearrow \tau$, so that $E(\tau \wedge k) \to E(\tau)$ by the monotone convergence theorem, while $S_0(\tau \wedge k) \to S_0(\tau)$ with $|S_0(\tau \wedge k)| \leq a \vee b < \infty$ for all k, and hence $E(S_0^2(\tau \wedge k)) \to E(S_0^2(\tau))$ by the dominated convergence theorem. Thus taking limits across (3) yields

$$E(S_0^2(\tau)) = E(\tau)\,,$$

and when $\mu = 0$, this implies that $E(\tau) < \infty$. By playing this game with the martingale S, we find that $E(S(\tau \wedge k)) = 0$, and by the dominated convergence theorem, $E(S(\tau)) = 0$. Since $S(\tau)$ takes on the two values $-a$ and b, we have

$$0 = E_0 S(\tau) = -aP_0(S(\tau) = -a) + bP_0(S(\tau) = b) = -a(1 - p_b) + bp_b$$

so that $p_b = a/(b+a)$, $p_a = 1 - p_b = b/(b+a)$. From $E(S_0^2(\tau)) = E(\tau)$ it then follows that

$$E(\tau) = a^2 p_a + b^2 p_b = a^2 \frac{b}{b+a} + b^2 \frac{a}{b+a} = ab\,,$$

completing the proof of Claim 2.
Proof of Claim 3. Similarly, when $\mu \neq 0$, the basic optional sampling theorem yields

$$0 = E(S_\mu(\tau \wedge k) - (\tau \wedge k)\mu)\,. \tag{4}$$

Now $\tau \wedge k \nearrow \tau$, so that $E(\tau \wedge k) \to E(\tau)$ by the monotone convergence theorem, while $S_\mu(\tau \wedge k) \to S_\mu(\tau)$ with $|S_\mu(\tau \wedge k)| \leq a \vee b < \infty$ for all k, and hence $E(S_\mu(\tau \wedge k)) \to E(S_\mu(\tau))$ by the dominated convergence theorem. Thus taking limits across (4) yields

$$E(S_\mu(\tau)) = \mu E(\tau)\,,$$

and this implies that $E(\tau) < \infty$ for $\mu \neq 0$. Again the basic optional sampling theorem implies that

$$E(Y(0)) = 1 = E \exp(-2\mu S_\mu(\tau \wedge k))\,,$$

for each k, and by the dominated convergence theorem this yields

$$\begin{aligned}
E(Y(0)) = 1 &= E\exp(-2\mu S_\mu(\tau)) \\
&= P(S_\mu(\tau) = -a)\exp(2\mu a) + P(S_\mu(\tau) = b)\exp(-2\mu b) \\
&= p_a \exp(2\mu a) + (1 - p_a)\exp(-2\mu b) \\
&= p_a(\exp(2\mu a) - \exp(-2\mu b)) + \exp(-2\mu b)
\end{aligned}$$

so that

$$p_a = \frac{1 - \exp(-2\mu b)}{\exp(2\mu a) - \exp(-2\mu b)} = \frac{1 - \exp(2\mu b)}{1 - \exp(2\mu(a+b)} .$$

Then, finally, since $E(S_\mu(\tau)) = \mu$,

$$\begin{aligned}
E(\tau) &= \frac{E(S_\mu(\tau))}{\mu} \\
&= \frac{1}{\mu}\{-ap_a + b(1 - p_a)\} \\
&= \frac{1}{\mu}\left\{ b - (a+b)\frac{1 - \exp(2\mu b)}{1 - \exp(2\mu(a+b)} \right\} .
\end{aligned}$$

Note that when $\mu < 0$ we have

$$\begin{aligned}
P(\|S_\mu^+\|_0^\infty \ge b) &= P(\sup_{0 \le t < \infty} S_\mu(t) \ge b) = \lim_{a\to\infty} P(\tau_{ab} < \infty) \\
&= \lim_{a\to\infty} P(S_\mu(\tau_{ab}) = b) \\
&= \exp(-2|\mu|b)
\end{aligned}$$

so that $\|S_\mu^+\|_0^\infty \sim \text{Exponential}(2|\mu|)$.

4. Redo problem 6 from Problem Set #4 for yourself, not relying on the solution set, and *doing it under the assumption that* $\{S_k\}$ *is a martingale* with $E(X_k^2) < \infty$ for each k with $X_k \equiv S_k - S_{k-1}$. The X_k's need not be independent!

   Solution: If $\{S_k, \mathcal{A}_k\}_{0 \le k \le N}$ is a 0−mean martingale and $X_k \equiv S_k - S_{k-1}$ has $Var(X_k) = \sigma_k^2 < \infty$ for all $1 \le k \le N$, then with

$$T_k \equiv S_n/b_n + \sum_{j=n+1}^{k} X_j/b_j, \qquad \text{for } \ n \le k \le N,$$

$\{T_k, \mathcal{A}_k\}_{n\le k\le N}$ is a martingale. To see this, note that

$$\begin{aligned}
E\{T_{k+1}|\mathcal{A}_k\} &= E\{S_n/b_n + \sum_{j=n+1}^{k+1}(X_j/b_j)|\mathcal{A}_k\} \\
&= S_n/b_n + \sum_{j=n+1}^{k}(X_j/b_j) + E\{(X_{k+1}/b_{k+1})|\mathcal{A}_k\} \\
&= T_k \quad \text{a.s.}
\end{aligned}$$

since $E(X_{k+1}|\mathcal{A}_k\} = E(S_{k+1}|\mathcal{A}_k\} - S_k = 0$ almost surely. Furthermore,

$$Var(T_N) = b_n^{-2}\sum_{k=1}^{n}\sigma_k^2 + \sum_{j=n+1}^{N}(\sigma_j^2/b_j^2)$$

since

$$\begin{aligned}
Var(T_N) &= Var(S_n/b_n) + 2b_n^{-1}Cov(S_n, \sum_{j=n+1}^{N}(X_j/b_j)) + Var(\sum_{j=n+1}^{N}X_j/b_j) \\
&= b_n^{-2}Var(S_n) + 0 + Var(\sum_{j=n+1}^{N}X_j/b_j) \\
&= b_n^{-2}\sum_{k=1}^{n}\sigma_k^2 + \sum_{j=n+1}^{N}\sigma_j^2/b_j^2
\end{aligned}$$

via the following additional calculations:

$$\begin{aligned}
Var(S_n) &= ES_n^2 = EE\{(S_{n-1}+X_n)^2|\mathcal{A}_{n-1}\} \\
&= E\{S_{n-1}^2 + 0 + E(X_n^2|\mathcal{A}_{n-1})\} \\
&= Var(S_{n-1}) + Var(X_n) = Var(S_{n-1}) + \sigma_n^2 \\
&= \cdots = \sum_{j=1}^{n}\sigma_j^2,
\end{aligned}$$

while, similarly,

$$
\begin{aligned}
Var(\sum_{j=n+1}^{N} X_j/b_j) &= E\left\{\left(\sum_{n+1}^{N} X_j/b_j\right)^2\right\} \\
&= EE\left\{\left(\sum_{n+1}^{N-1} X_j/b_j\right)^2 + 2b_N^{-1}X_N\left(\sum_{n+1}^{N-1} X_j/b_j\right) + b_N^{-2}X_N^2|\mathcal{A}_N\right\} \\
&= Var(\sum_{j=n+1}^{N-1} X_j/b_j) + 0 + b_N^{-2}\sigma_N^2 \\
&= \cdots = \sum_{j=n+1}^{N} \sigma_j^2/b_j^2.
\end{aligned}
$$