Statistics 522, Problem Set 4 Solutions

Wellner; 2/13/2013

Reminder: Midterm Exam, Friday, February 15.

1. Exercise 13.3.6, PfS Course Notes, page 359. [Exercise 18.3.5, PfS (2000), page 477.] Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a sub-martingale with $X_n \ge 0$. Let $r > 1$. Then $\{X_n^r\}$ is uniformly integrable if and only if $\{X_n^r\}$ is integrable.

   Solution: Uniform integrability implies integrability, so it remains only to prove the reverse implication. Suppose that $\{X_n^r\}$ is integrable. Then $\{X_n\}$ is uniformly integrable, and hence by the s-martingale convergence theorem 18.3.1, $X_n \to X_\infty \in L_1$ where $\{X_n, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a sub-mg; i.e. $E(X_\infty|\mathcal{A}_n) \ge X_n$ a.s. and

   $$E(X_\infty^r) = E(\liminf X_n^r) \le \liminf E(X_n^r) \le \sup_n E(X_n^r) < \infty$$

   by Fatou's lemma and integrability of $\{X_n^r\}$. Hence by the conditional Jensen inequality,

   $$E(X_n^r) \le E\{E(X_\infty|\mathcal{A}_n)^r\} \le E\{E(X_\infty^r|\mathcal{A}_n)\} = E(X_\infty^r)$$

   and it follows from Vitali's theorem that $\{X_n^r\}$ is uniformly integrable.

   Alternatively, by Doob's L_r-maximal inequality, since $\{X_n, \mathcal{A}_n\}$ is a sub-martingale,

   $$E\left\{\left(\max_{1\le k\le n} X_k\right)^r\right\} \le \left(\frac{r}{r-1}\right)^r E|X_n|^r\,,$$

   and hence, by the monotone convergence theorem,

   $$E[\sup_{1\le k<\infty} X_k^r] \le \left(\frac{r}{r-1}\right)^r \sup_n E|X_n|^r < \infty\,.$$

   Thus with $Y \equiv \sup_{1\le k<\infty} X_k$, it follows that

   $$\sup_n E\left\{X_n^r 1_{[X_n^r\ge\lambda]}\right\} \le E(Y^r 1_{[Y^r\ge\lambda]} \to 0$$

   as $\lambda \to \infty$; i.e. $\{X_n^r\}$ is uniformly integrable.

2. Exercise 13.3.7, PfS Course Notes, page 359. [Exercise 18.3.7, PfS (2000), page 477.] Let $r > 1$. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a martingale. Then the following are equivalent:
   (10) The $|X_n|^r-$process is integrable.
   (11) $X_n \to_r X_\infty$
   (12) The X_n's are uniformly integrable (thus $X_n \to$ (some X_∞) a.s.) and $X_\infty \in L_r$.

(13) The $|X_n|^r$'s are uniformly integrable.
(14) $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\infty}$ is a submg and $E|X_n|^r \nearrow E|X_\infty|^r$.

Solution: Suppose that (10) holds. Then $|X_n|^r$ is an integrable sub-mg. Thus the $|X_n|^r$ are uniformly integrable by the preceding problem. Thus (13) holds.
Suppose (13) holds. Then $\{X_n\}$ is uniformly integrable, and $X_n \to_{a.s.} X_\infty \in L_1$ and

$$E|X_\infty|^r = E(\liminf |X_n|^r) \le \liminf E|X_n|^r \le \sup_n E|X_n|^r < \infty\,,$$

so $X_\infty \in L_r$; i.e. (12) holds.
Suppose (12) holds. Then $\{|X_n|, \mathcal{A}_n\}_{n=0}^{\infty}$ is a sub-martingale by Theorem 16.3.1. Thus $|X_n| \le E(|X_\infty|\mathcal{A}_n)$, so $|X_n|^r \le \{E(|X_\infty||\mathcal{A}_n)\}^r \le E(|X_n|^r|\mathcal{A}_n)$ a.s., and hence $E|X_n|^r \le E|X_\infty|^r < \infty$; i.e. (10) holds.
Thus (10) iff (12) iff (13) holds.
Now (11) implies (10) since

$$E|X_n|^r \le c_r\{E|X_n - X_\infty|^r + E|X_\infty|^r\}$$

by the c_r-inequality.
Suppose that (13) holds. Then $X_n \to_{a.s.} X_\infty \in L_r$ (by (13) implies (12)), and since $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\infty}$ is a sub-mg,

$$\limsup_{n\to\infty} E|X_n|^r \le E|X_\infty|^r < \infty\,.$$

Hence $X_n \to_r X_\infty$ by Vitali's theorem; i.e. (11) holds. Thus (10) iff (12) iff (13) iff (14).

3. Exercise 13.4.3, PfS Course Notes, page 365. (Conditional Borel-Cantelli) Let $\mathcal{A}_n$ be an increasing sequence of $\sigma-$fields in $\mathcal{A}$, and let $A_n \in \mathcal{A}_n$. Show that $[A_n\ i.o.] = [\omega : \sum_{n=1}^{\infty} P(A_n|\mathcal{A}_{n-1})(\omega) = \infty]$ almost surely.

Solution: To ease notation slightly, I will let the events be called B_n rather than A_n. Then we want to show that $[B_n\ i.o.] = [\omega : \sum_{k=1}^{\infty} P(B_k|\mathcal{A}_{k-1})(\omega) = \infty]$ almost surely. Let $X_0 \equiv 0$ and let

$$X_n \equiv \sum_{k=1}^{n} \{1_{B_k} - P(B_k|\mathcal{A}_{k-1}) = \sum_{k=1}^{n} 1_{B_k} - \sum_{k=1}^{n} P(B_k|\mathcal{A}_{k-1}) \equiv Z_n - Y_n$$

for $n \ge 1$. Then X_n is a martingale with bounded increments: $|X_n - X_{n-1}| \le 1$; hence $EX_n^2 < \infty$ for all $n \ge 0$. With this notation our goal is to show that $[Z_\infty = \infty] = [B_n\ i.o.] = [Y_\infty = \infty]$. Note that

$$\langle X\rangle_n \;=\; \sum_{k=1}^{n} E\{(\Delta X_k)^2|\mathcal{A}_{k-1}\} = \sum_{k=1}^{n} E\{(1_{B_k} - P(B_k|\mathcal{A}_{k-1}))^2|\mathcal{A}_{k-1}\} \qquad (0.1)$$

$$= \sum_{k=1}^{n} P(B_k|\mathcal{A}_{k-1})(1 - P(B_k|\mathcal{A}_{k-1})) \quad (0.2)$$

$$\leq \sum_{k=1}^{n} P(B_k|\mathcal{A}_{k-1}) = Y_n \nearrow Y_\infty. \quad (0.3)$$

If $Y_\infty(\omega) < \infty$, then $\langle X\rangle_\infty(\omega) < \infty$, and hence by (a) of the martingale strengthening of the 2-series theorem proved in class (on 2/11/13), $\lim_n X_n(\omega)$ exists and is finite, so $Z_\infty(\omega) < \infty$. Thus $Z_\infty(\omega) = \infty$ implies $Y_\infty(\omega) = \infty$, and hence $[Z_\infty = \infty] \subset [Y_\infty = \infty]$. I do not have a separate proof of the reverse inclusion yet. But here is a proposition in a similar spirit to Theorem 5.3.1 of Durrett (200x), page 239, which yields the desired conclusion.

Proposition: Let $\{X_n, \mathcal{A}_n\}_{n\geq 0}$ be a martingale with $|X_{n+1} - X_n| \leq M$ almost surely. let

$$G \equiv [\lim_n X_n \text{ exists and is finite}],$$
$$H \equiv [\limsup_n X_n = +\infty,\ \liminf X_n = -\infty].$$

Then $P(G \cup H) = 1$.

Proof. Without loss of generality we can suppose $X_0 = 0$. Let $K > 0$ and let $T \equiv \inf\{n : X_n \geq K\}$. Now $\{X_{n\wedge T}, \mathcal{A}_{n\wedge T}\}_{n\geq 0}$ is a martingale with $X_{n\wedge T} \leq K + M$ a.s., so $E(X^+_{n\wedge T}) \leq E(X_{n\wedge T}) \leq K + M$, and hence the $s-$martingale convergence theorem applies: $X_{n\wedge T} \to \text{some} X_\infty \in L_1(P)$. Thus $\lim_n X_n$ exists on $[T = \infty]$. Letting $K \to \infty$ shows that $\lim_n X_n$ exists on $\limsup_n X_n < \infty$. Applying this last conclusion to $\{-X_n\}$ shows that $\lim_n X_n$ exists on $\liminf_n X_n > -\infty$. This yields the claim. □

Now we apply this to the martingale $\{X_n, \mathcal{A}_n\}$ of the current problem:
on G, $Z_\infty = \infty$ if and only if $Y_\infty = \infty$.
on H, both $Z_\infty = \infty$ and $Y_\infty = \infty$. Since $P(G \cup H) = 1$, the claimed result follows: $[Z_\infty = \infty] = [Y_\infty = \infty]$ almost surely.

4. Exercise 13.4.4, PfS Course Notes, page 366. [Exercise 18.4.3, PfS (2000), page 484.]

Solution: With $Z_{n+1}\ \sum_{j=1}^{Z_n} X_{n,j}$, $Z_0 \equiv 1$, and all $X_{n,j}$ i.i.d. as X with $E(X) = m$, we know that $W_n \equiv Z_n/m^n$ is a mean 1 martingale. Thus $\{W_n^2, \mathcal{A}_n\}_{n\geq 0}$ is a submartingale (assuming $EX^2 < \infty$), and $\{W_n^2 - \langle W\rangle_n, \mathcal{A}_n\}_{n\geq 0}$ is a martingale where

$$\langle W\rangle_n = \sum_{k=1}^{n} E\{(\Delta W_k)^2|\mathcal{A}_{k-1}\} + EW_0^2$$

and we have

$$E(W_n^2) = E\langle W\rangle_n = \sum_{k=1}^{n} Var(\Delta W_k) + EW_0^2.$$

Since $EW_0^2 = 1 = (EW_0)^2 = (EW_n)^2$ for all n, this yields

$$Var(W_n) = \sum_{k=1}^{n} Var(\Delta W_k).$$

But since $E(\Delta W_k | \mathcal{A}_{k-1}) = 0$ a.s.,

$$Var(\Delta W_k) = EVar(\Delta W_k | \mathcal{A}_{k-1})$$

where

$$\begin{aligned}
Var(\Delta W_k | \mathcal{A}_{k-1}) &= E\{W_k^2 | \mathcal{A}_{k-1}\} - W_{k-1}^2 \\
&= E\{\left(\frac{Z_k}{m^k}\right)^2 | \mathcal{A}_{k-1}\} - W_{k-1}^2 \\
&= \frac{1}{m^{2k}}\{Var(Z_k|\mathcal{A}_{k-1}) + E(Z_k|\mathcal{A}_{k-1})^2\} - W_{k-1}^2 \\
&= \frac{1}{m^{2k}}\{Z_{k-1}\sigma^2 + (Z_{k-1}m)^2\} - W_{k-1}^2 \\
&= \frac{\sigma^2}{m^{k+1}} W_{k-1} + W_{k-1}^2 - W_{k-1}^2 \\
&= \frac{\sigma^2}{m^{k+1}} W_{k-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
Var(W_n) &= \sum_{k=1}^{n} E\left\{\frac{\sigma^2}{m^{k+1}} W_{k-1}\right\} = \sum_{k=1}^{n} \frac{\sigma^2}{m^{k+1}} \\
&= \begin{cases} n\sigma^2, & \text{if } m = 1, \\ \sigma^2 \frac{1-m^{-n}}{m(m-1)}, & \text{if } m \neq 1 \end{cases}
\end{aligned}$$

where we used (for $m > 1$)

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{m^{k+1}} &= \frac{1}{m^2} \sum_{j=0}^{n-1} \frac{1}{m^j} = \frac{1}{m^2} \left\{\sum_{j=0}^{\infty} \frac{1}{m^j} - \sum_{j=n}^{\infty} \frac{1}{m^j}\right\} \\
&= \frac{1}{m^2} \left\{\frac{1}{1-1/m} - \frac{1}{m^n} \frac{1}{1-1/m}\right\} \\
&= \frac{1-m^{-n}}{m(m-1)}.
\end{aligned}$$

5. Suppose that $X_1, X_2, \ldots$ are independent random variables on $(\Omega, \mathcal{A})$ and that X_n has density p_n or q_n under P or Q respectively where p_n and q_n are (for simplicity) everywhere positive on $\mathbb{R}$. Let $\mathcal{F} = \sigma[X_1, X_2, \ldots]$ and $\mathcal{F}_n = \sigma[X_1, \ldots, X_n]$ for $n \geq 1$.

Let $Y_n \equiv q_n(X_n)/p_n(X_n)$.
(a) Show that

$$M_n \equiv \frac{dQ}{dP}\Big|_{\mathcal{F}_n} = Y_1 \cdots Y_n$$

where the Y_n's are independent and have mean 1 under P; Hence the likelihood ratio martingale of Example 1.14 is the Kakutani product martingale of Example 1.15.
(b) Show that Q is absolutely continuous relative to P on $\mathcal{F}$ if and only if the martingale $\{M_n, \mathcal{F}_n\}$ is uniformly integrable.
(c) Conclude from Kakutani's theorem (PfS Example 4.4, pages 482-483) that $Q \ll P$ on $\mathcal{F}$ if and only if

$$\prod_{n=1}^{\infty} E(Y_n^{1/2}) = \prod_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{p_n(x) q_n(x)}dx > 0\,.$$

(d) Construct two examples of sequences p_n and q_n, one in which the condition in (c) holds and one in which it fails. What is the statistical meaning when it holds and when it fails?

Solution: (a) Let $A_i \in \sigma(X_i)$ for $i = 1, \ldots, n$. Then

$$\begin{aligned}
E_P\{1_{A_1\times\cdots\times A_n}\frac{dQ}{dP}\} &= E_Q\{1_{A_1\times\cdots\times A_n}\} \\
&\qquad \text{by definition of the Radon-Nikodym derivative} \\
&= \prod_{i=1}^{n} E_Q(1_{A_i}) \qquad \text{by independence} \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} 1_{A_i}(x) q_i(x) d\mu(x) \qquad \text{by existence of the densities } q_i \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} 1_{A_i}(x) \frac{q_i(x)}{p_i(x)} p_i(x) d\mu(x) \\
&= \prod_{i=1}^{n} E_P\{1_{A_i} Y_i\} \\
&= E_P\{1_{A_1\times\cdots\times A_k} Y_1 \cdots Y_n\} \qquad \text{by independence}\,.
\end{aligned}$$

Now $Y_1 \cdots Y_n$ is $\mathcal{F}_n$ measurable (since it is a function of $X_1, \ldots, X_n$ and agrees with $dQ/dP|_{\mathcal{F}_n}$ on the $\overline{\pi}-$ system $\sigma(X_1) \times \cdots \times \sigma(X_n)$. Thus the claimed equality holds. The Y_i's are independent because the X_i's are independent and they have mean 1 because

$$E_P Y_i = \int_{\mathbb{R}} \frac{q_i(x)}{p_i(x)} p_i(x) d\mu(x) = \int_{\mathbb{R}} q_i(x) dx = 1\,.$$

(b) If $Q \ll P$, with Radon-Nikodym derivative $dQ/dP \equiv Z$, then $M_n = E(Z|\mathcal{F}_n)$

with $E_P(Z) = Q(\mathbb{R}^\infty) = 1$, so $\{M_n, \mathcal{F}_n\}_{n=0}^{\overline{\infty}}$ is a martingale closed at infinity and is uniformly integrable. Conversely, if $\{M_n\}$ is uniformly integrable, then $M_n \to_{a.s.} M_\infty$ and $E(M_\infty | \mathcal{F}_n) = M_n$ almost surely for every n. Now consider the measures Q and $\tilde{Q}$ defined by

$$\tilde{Q}(A) = E\{1_A M_\infty\}\,.$$

These measures agree on the π-system $\cup \mathcal{F}_n$, and hence they agree on $\mathcal{F} = \sigma(\cup \mathcal{F}_n)$. This implies (by the pi-lambda theorem) that Q and $\tilde{Q}$ agree on $\mathcal{F}$, and hence $M_\infty = dQ/dP$ on $\mathcal{F}$, and $Q \ll P$.

(c) By Kakutani's theorem we conclude that Q is absolutely continuous with respect to P on $\mathcal{F}$ if and only if

$$\prod_{n=1}^\infty E(Y_n^{1/2}) = \prod_{n=1}^\infty \int_{\mathbb{R}} (q_n(x)/p_n(x))^{1/2} p_n(x)dx = \prod_{n=1}^\infty \int_{\mathbb{R}} \sqrt{p_n(x) q_n(x)} dx > 0\,.$$

Since this is symmetric in p_n and q_n and since these densities are everywhere positive, we also can conclude that P is absolutely continuous with respect to Q on $\mathcal{F}$; thus Q and P are mutually absolutely continuous or *equivalent* on $\mathcal{F}$.

(d) Suppose that $p_n(x) = \exp(-x) 1_{[0,\infty)}(x)$ and $q_n(x) = \lambda_n \exp(-\lambda_n x) 1_{[0,\infty)}(x)$ with $\lambda_n = 1 + c_n$ where $c_n \to 0$. Then we compute

$$E(Y_n^{1/2}) = \int_0^\infty \lambda_n^{1/2} \exp(-(1+\lambda_n)x/2) dx = \frac{2\lambda_n^{1/2}}{1+\lambda_n}\,,$$

and

$$\begin{aligned}
H^2(P_n, Q_n) &= \frac{1}{2} \int (\sqrt{p_n(x)} - \sqrt{q_n(x)})^2 dx = 1 - E(Y_n^{1/2}) \\
&= 1 - \frac{2\lambda_n^{1/2}}{1+\lambda_n} \\
&= \frac{1 + \lambda_n - 2\lambda_n^{1/2}}{1+\lambda_n} \\
&= \frac{2 + c_n - 2(1+c_n)^{1/2}}{2 + c_n} \\
&\sim \frac{1}{8} c_n^2 \qquad \text{as} \ \ n \to \infty
\end{aligned}$$

since $c_n \to 0$ and $(1+c_n)^{1/2} = 1 + (1/2)c_n - (1/8 + o(1))c_n^2$. Thus if $c_n = n^{-r}$ with $r > 1/2$ it follows that

$$\sum_1^\infty (1 - E(Y_n^{1/2})) = \sum_1^\infty H^2(P_n, Q_n) < \infty\,,$$

and $Q \ll P$ on $\mathcal{F}$. If $c_n = n^{-1/2}$, then

$$\sum_1^\infty (1 - E(Y_n^{1/2})) = \sum_1^\infty H^2(P_n, Q_n) = \infty\,,$$

and by Kakutuni's theorem we conclude that $M_\infty = 0$ almost surely P. In this case Q and P are singular on $\mathbb{R}^\infty$: there is a set $A \subset \mathbb{R}^\infty$ such that $Q(A) = 1$ and $P(A) = 0$; i.e. $P(A^c) = 1$.