

## Statistics 522, Problem Set 3, Solutions

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1. Exercise 12.4.1, page 313, PfS Course Notes, Chapter 12.  
(Exercise 12.4.1, page 306, PfS, 2000.)

**Solution:** (5) Let  $T_1$  and  $T_2$  be (extended) stopping times. To see that  $T_1 + T_2$  is an extended stopping time in the discrete time case (when  $T_1, T_2 \in \mathbb{N} \cup \{\infty\}$ ), note that

$$[T_1 + T_2 \leq n] = \cup_{k=0}^n [T_1 = k] \cap [T_2 = n - k]$$

where  $[T_1 = k] \in \mathcal{A}_k \subset \mathcal{A}_n$  and  $[T_2 = n - k] \in \mathcal{A}_{n-k} \subset \mathcal{A}_n$  for  $k = 0, \dots, n$ . Hence  $[T_1 + T_2 \leq n] \in \mathcal{A}_n$  and  $T_1 + T_2$  is an (extended) stopping time.

For the continuous case we note (from the hint) that

$$[T_1 + T_2 \leq u] = \cap_{m=1}^{\infty} \{ \cup_{a,b \in \mathbb{Q}, a+b \leq u} [T_1 \leq a + 1/m] \cap [T_2 \leq b + 1/m] \}$$

where

$$\begin{aligned} [T_1 \leq a + 1/m] &\in \mathcal{A}_{a+1/m} \subset \mathcal{A}_{a+b+2/m} \subset \mathcal{A}_{u+2/m}, \\ [T_2 \leq b + 1/m] &\in \mathcal{A}_{b+1/m} \subset \mathcal{A}_{a+b+2/m} \subset \mathcal{A}_{u+2/m}, \end{aligned}$$

for every  $m \geq 1$ . But this implies that  $[T_1 + T_2 \leq u] \in \mathcal{A}_{u+} = \mathcal{A}_u$ .

(6) Suppose  $A \in \mathcal{A}_{T_1}$ . But then

$$\begin{aligned} &A \cap [T_1 \leq T_2] \cap [T_2 \leq t] \\ &= A \cap [T_1 \leq t] \cap [T_1 \wedge t \leq T_2 \wedge t] \cap [T_2 \leq t] \\ &= \text{something in } \mathcal{A}_t \cap \text{something in } \mathcal{A}_t \cap \text{something in } \mathcal{A}_t \\ &\in \mathcal{A}_t. \end{aligned}$$

Hence  $A \cap [T_1 \leq T_2] \in \mathcal{A}_{T_2}$ .

To see that  $[T_1 < T_2]$ ,  $[T_1 = T_2]$ , and  $[T_1 > T_2]$  are all in  $\mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$ ,

first note that

$$\begin{aligned}
& [T_1 < T_2] \cap [T_2 \leq t] \\
&= [T_1 < t] \cap [T_2 \leq t] \cap [T_1 \wedge t < T_2 \wedge t] \\
&= \bigcup_{n=1}^{\infty} [T_1 \leq t - 1/n] \cap [T_2 \leq t] \cap [T_1 \wedge t < T_2 \wedge t] \\
&= \bigcup_{n=1}^{\infty} \text{something in } \mathcal{A}_{t-1/n} \cup \text{something in } \mathcal{A}_t \cap \text{something in } \mathcal{A}_t \\
&\in \mathcal{A}_t,
\end{aligned}$$

so that  $[T_1 < T_2] \in \mathcal{A}_{T_2}$ . Then by taking  $A = \Omega$  in the first part of the proof yields  $[T_1 \leq T_2] \in \mathcal{A}_{T_2}$ , and hence also  $[T_1 = T_2] = [T_1 < T_2]^c \cap [T_1 \leq T_2] \in \mathcal{A}_{T_2}$ . By symmetry we have  $[T_1 = T_2] \in \mathcal{A}_{T_1}$  as well. And by symmetry again  $[T_2 < T_1] \in \mathcal{A}_{T_1}$  and hence also  $[T_2 \leq T_1] = [T_2 < T_1] \cup [T_2 = T_1] \in \mathcal{A}_{T_1}$ . This yields  $[T_1 < T_2] = [T_2 \leq T_1]^c \in \mathcal{A}_{T_1}$ .

(7) If  $A \in \mathcal{A}_{T_1}$ , then  $A \cap [T_2 \leq t] = (A \cap [T_1 \leq t]) \cap [T_2 \leq t] \in \mathcal{A}_t$  for all  $t \geq 0$  since  $T_1 \leq T_2$ , and hence  $A \in \mathcal{A}_{T_2}$ .

Also, for  $A \in \mathcal{A}_{T_1}$ ,

$$\begin{aligned}
& A \cap [T_1 \leq T_2] \cap [T_1 \wedge T_2 \leq t] \\
&= A \cap [T_1 \leq T_2] \cap ([T_1 \leq t] \cup [T_2 \leq t]) \\
&= (A \cap [T_1 \leq T_2]) \cap [T_1 \leq t] \cup (A \cap [T_1 \leq T_2]) \cap [T_2 \leq t] \\
&= \text{something in } \mathcal{A}_{T_1} \cap [T_1 \leq t] \cup \text{something in } \mathcal{A}_{T_2} \cap [T_2 \leq t] \\
&= \text{something in } \mathcal{A}_t \cup \text{something in } \mathcal{A}_t \\
&\in \mathcal{A}_t.
\end{aligned}$$

for all  $t \geq 0$ , so  $\mathcal{A}_{T_1} \cap [T_1 \leq T_2] \subset \mathcal{A}_{T_1 \wedge T_2}$ .

To see that  $\mathcal{A}_{T_1 \wedge T_2} = \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$ , suppose first that  $A \in \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$ .

Then

$$\begin{aligned}
A \cap [T_1 \wedge T_2 \leq t] &= [A \cap [T_1 \wedge T_2 \leq t] \cap ([T_1 \leq T_2] \cup [T_1 > T_2])] \\
&= ((A \cap [T_1 \leq T_2]) \cap [T_1 \leq t]) \cup (A \cap [T_1 > T_2]) \cap [T_2 \leq t] \\
&= \text{something in } \mathcal{A}_{T_1} \cap [T_1 \leq t] \cup \text{something in } \mathcal{A}_{T_2} \cap [T_2 \leq t] \\
&\quad \text{by using } [T_1 \leq T_2] \in \mathcal{A}_{T_1} \text{ and } [T_1 > T_2] \in \mathcal{A}_{T_2} \\
&\in \text{something in } \mathcal{A}_t \cup \text{something in } \mathcal{A}_t \\
&\in \mathcal{A}_t.
\end{aligned}$$

For the reverse inclusion, suppose that  $A \in \mathcal{A}_{T_1 \wedge T_2}$ . But by the first part of the problem  $T_1 \wedge T_2 \leq T_j$  for  $j = 1, 2$ , and hence  $\mathcal{A}_{T_1 \wedge T_2} \subset \mathcal{A}_{T_j}$

for  $j = 1, 2$ . Thus  $A \in \mathcal{A}_{T_1} \cap \mathcal{A}_{T_2}$ .

(8) Since  $T_0 \leq T_n$  for all  $n$ , it follows from (7) that  $\mathcal{A}_{T_0} \subset \mathcal{A}_{T_n}$  for all  $n$  and hence  $\mathcal{A}_{T_0} \subset \bigcap_{n=1}^{\infty} \mathcal{A}_{T_n}$ . To show the reverse inclusion, let  $A \in \bigcap_{n=1}^{\infty} \mathcal{A}_{T_n}$ . Since  $A \cap [T_n \leq t] \in \mathcal{A}_t$  for all  $t \geq 0$  and  $T_n \searrow T_0$ , it follows that  $A \cap [T_0 \leq t] \in \mathcal{A}_t$  for all  $t \geq 0$ , and hence  $A \in \mathcal{A}_{T_0}$ .

2. Exercise 13.1.4, page 353, PFS Course Notes, Chapter 13.  
(Exercise 18.1.4, page 471, PFS, 2000.)

**Solution:** Let  $X \sim F$  on  $\mathbb{R}$ , let  $\mathbb{N}_t \equiv 1_{[X \leq t]}$ , and let  $\mathcal{A}_t \equiv \sigma[\mathbb{N}_r : r \leq t]$ . First we show that for  $s < t$  we have

$$E\{\mathbb{N}_t | \mathcal{A}_s\} = 1_{[X \leq s]} + 1_{[X > s]} \frac{F(s, t)}{1 - F(s)} \quad \text{a.s.}$$

where  $F(s, t) \equiv F(t) - F(s)$ . To see this, note that it holds if we can show that

$$E \left[ 1_A \left\{ \mathbb{N}_t - \left( 1_{[X \leq s]} + 1_{[X > s]} \frac{F(s, t)}{1 - F(s)} \right) \right\} \right] = 0 \quad (1)$$

for each set  $A \in \mathcal{A}_s$ . But since  $\mathcal{C}_s \equiv \{[X > r] : -\infty \leq r \leq s\}$  is a  $\bar{\pi}$ -system that generates  $\mathcal{A}_s$ , it suffices to show that (1) holds for each  $A$  of the form  $[X > r]$  with  $r \leq s$ . But we compute

$$\begin{aligned} & E\{1_{[X > r]} \left( \mathbb{N}_t - \left( 1_{[X \leq s]} + 1_{[X > s]} \frac{F(s, t)}{1 - F(s)} \right) \right) \\ &= E \left\{ 1_{[r < X \leq t]} - 1_{[r < X \leq s]} - 1_{[X > s]} \frac{F(s, t)}{1 - F(s)} \right\} \\ &= F(r, t) - F(r, s) - F(s, t) = F(r, t) - F(r, t) = 0. \end{aligned}$$

Thus (1) holds. Similarly, to show that

$$E\{1_{[X \geq u]} | \mathcal{A}_s\} = 1_{[X > s]} \frac{1 - F(u-)}{1 - F(s)} \quad \text{for } -\infty < s < u \leq t < \infty$$

it suffices to show that

$$E \left\{ 1_A \left( 1_{[X \geq u]} - 1_{[X > s]} \frac{1 - F(u-)}{1 - F(s)} \right) \right\} = 0 \quad (2)$$

for all  $A$  of the form  $[X > r]$  with  $r \leq s$ . To see this we compute

$$\begin{aligned} E \left\{ 1_{[X > r]} \left( 1_{[X \geq u]} - 1_{[X > s]} \frac{1 - F(u-)}{1 - F(s)} \right) \right\} &= E \left\{ 1_{[X \geq u]} - 1_{[X > s]} \frac{1 - F(u-)}{1 - F(s)} \right\} \\ &= 1 - F(u-) - (1 - F(u)) = 0, \end{aligned}$$

and hence (2) holds. Finally, with

$$\mathbb{M}_t \equiv \mathbb{N}_t - \int_{(-\infty, t]} 1_{[X \geq r]} d\Lambda(r) \equiv \mathbb{N}_t - \mathbb{A}_t$$

we have first, for  $r \leq s < t$ ,

$$\begin{aligned} E\{1_{[X > r]}(\mathbb{N}_t - \mathbb{N}_s)\} &= E\{1_{[r < X \leq t]} - 1_{[r < X \leq s]}\} \\ &= F(r, t] - F(r, s] = F(s, t], \end{aligned}$$

and, on the other hand

$$\begin{aligned} E\{1_{[X > r]}(\mathbb{A}_t - \mathbb{A}_s)\} &= E \left\{ 1_{[X > r]} \int_{(s, t]} 1_{[X \geq v]} d\Lambda(v) \right\} \\ &= \int_{(s, t]} E\{1_{[X > r]} 1_{[X \geq v]}\} d\Lambda(v) \\ &= \int_{(s, t]} E\{1_{[X \geq v]}\} d\Lambda(v) \\ &= \int_{(s, t]} (1 - F(v-)) \frac{1}{1 - F(v-)} dF(v) \\ &= F(s, t]. \end{aligned}$$

Thus

$$E\{1_A(\mathbb{M}_t - \mathbb{M}_s)\} = 0$$

for each  $A \in \mathcal{C}_s$  and hence for each  $A \in \mathcal{A}_s$ . Thus  $\{\mathbb{M}_t, \mathcal{A}_t\}$  is a martingale.

3. Exercise 13.1.5, page 353, PfS Course Notes, Chapter 13.  
(Exercise 18.1.5, page 471, PfS, 2000.)

**Solution:** As in Example 1.11,

$$\mathbb{N}_n(t) \equiv n\mathbb{G}_n(t) = \#\{i \leq n : \xi_i \leq t\} = \sum_{i=1}^n 1_{[0, t]}(\xi_i).$$

Then with  $\mathcal{A}_n(t) \equiv \sigma[1_{[\xi_i \leq s]} : 0 \leq s \leq t, 1 \leq i \leq n]$ , and

$$\mathbb{M}_n(t) \equiv \mathbb{N}_n(t) - \int_{[0,t]} \frac{n(1 - \mathbb{G}_n(r-))}{1-r} dr,$$

$\{\mathbb{M}_n(t), \mathcal{A}_n(t)\}_{0 \leq t < 1}$  is a martingale for each  $n$ . To see this, note that

$$\mathbb{M}_n(t) = \sum_{i=1}^n \left\{ 1_{[\xi_i \leq t]} - \int_{[0,t]} 1_{[\xi_i \geq r]} \frac{1}{1-r} dr \right\}$$

where the terms in the sum are independent, and hence it suffices to establish the claim for  $n = 1$ . Here is a detailed calculation: for any fixed  $n \geq 1$  and  $0 \leq s < t \leq 1$  we have:

$$\begin{aligned} E\{\mathbb{M}_n(t) | \mathcal{A}_n(s)\} &= \sum_{i=1}^n E \left\{ 1_{[\xi_i \leq t]} - \int_{[0,t]} 1_{[\xi_i \geq r]} \frac{1}{1-r} dr \middle| \mathcal{A}_n(s) \right\} \\ &= \sum_{i=1}^n E \left\{ 1_{[\xi_i \leq t]} - \int_{[0,t]} 1_{[\xi_i \geq r]} \frac{1}{1-r} dr \middle| \mathcal{A}_{n,i}(s) \right\} \\ &\quad \text{where } \mathcal{A}_{n,i}(s) \equiv \sigma[1_{[\xi_i \leq r]} : r \leq s], \quad 1 \leq i \leq n, \\ &= \sum_{i=1}^n \left\{ 1_{[\xi_i \leq s]} - \int_{[0,s]} 1_{[\xi_i \geq r]} \frac{1}{1-r} dr \right\} \quad \text{a.s.} \\ &= \mathbb{M}_n(s) \end{aligned}$$

where the next to last equality holds via the previous exercise. To see the second part of (16), note that

$$\begin{aligned} \mathbb{M}_n(t) &= \mathbb{N}_n(t) - nt + nt - \int_{[0,t]} \frac{n(1 - \mathbb{G}_n(r-))}{1-r} dr \\ &= \sqrt{n} \mathbb{U}_n(t) + \int_{[0,t]} \frac{n(1-r)}{1-r} dr - \int_{[0,t]} \frac{n(1 - \mathbb{G}_n(r-))}{1-r} dr \\ &= \sqrt{n} \mathbb{U}_n(t) + \int_{[0,t]} \frac{\mathbb{U}_n(r-)}{1-r} dr \end{aligned}$$

by using  $\mathbb{U}_n(t) \equiv \sqrt{n}(\mathbb{G}_n(t) - t)$ .

4. Exercise 13.1.6, page 353, Pfs Course Notes, Chapter 13.  
(Exercise 18.1.6, page 471, Pfs, 2000.)

**Solution:** In example 1.12,  $N(t)$  is a Poisson process with intensity  $\lambda$ ; and  $M(t) = N(t) - \lambda t$  and  $M^2(t) - \lambda t$  are both martingales. A natural exponential martingale to consider is

$$Y(t) \equiv Y_c(t) \equiv \frac{\exp(cM(t))}{E \exp(cM(t))}.$$

Since

$$\begin{aligned} E \exp(cM(t)) &= E \exp(c(N(t) - \lambda t)) \\ &= \exp(-c\lambda t) E \exp(cN(t)) \\ &= \exp(-c\lambda t) \exp((e^c - 1)\lambda t), \end{aligned}$$

we find that

$$Y(t) = \exp(cN(t) - (e^c - 1)\lambda t).$$

I claim that  $\{Y(t), \mathcal{A}_t\}_{t=0}^\infty$  is a martingale on  $[0, \infty)$ . To see this, note that for  $0 \leq s < t < \infty$  we have

$$\begin{aligned} E(Y(t)|\mathcal{A}_s) &= E(\exp(cN(t) - (e^c - 1)\lambda t)|\mathcal{A}_s) \\ &= E(\exp(c(N(t) - N(s)) - (e^c - 1)\lambda(t - s)) \exp(cN(s) - (e^c - 1)\lambda s)|\mathcal{A}_s) \\ &= Y(s) E(\exp(c(N(t) - N(s)) - (e^c - 1)\lambda(t - s))|\mathcal{A}_s) \quad \text{a.s.} \\ &= Y(s) E(\exp(c(N(t) - N(s)) - (e^c - 1)\lambda(t - s))) \quad \text{a.s.} \\ &\quad \text{since } N(t) - N(s) \text{ is independent of } \mathcal{A}_s \\ &= Y(s) \cdot 1 = Y(s) \quad \text{a.s. ,} \end{aligned}$$

and hence  $\{Y(t), \mathcal{A}_t\}_{t=0}^\infty$  is a martingale. Note that

$$\begin{aligned} Y'_c(t) \equiv \frac{d}{dc} Y_c(t) \Big|_{c=0} &= Y_c(t) (N(t) - e^c \lambda t) \Big|_{c=0} \\ &= Y_0(t) (N(t) - e^c \lambda t) \Big|_{c=0} \\ &= N(t) - \lambda t = M(t), \end{aligned}$$

while

$$\begin{aligned} Y''_c(t) \equiv \frac{d^2}{dc^2} Y_c(t) \Big|_{c=0} &= Y_c(t) (N(t) - e^c \lambda t)^2 \Big|_{c=0} + Y_c(t) (-e^c \lambda t) \Big|_{c=0} \\ &= M^2(t) - \lambda t. \end{aligned}$$

Further derivatives yield still more martingales.

5. Exercise 8.9.2, page 186: Turn  $\{S_k^2, \mathcal{A}_k\}_{1 \leq k \leq n}$  into a martingale by centering it appropriately.

**Solution:** By example 8.9.1, page 186,  $\{S_k^2, \mathcal{A}_k\}_{1 \leq k \leq n}$  is a sub-martingale. Furthermore,

$$\begin{aligned}
E\{S_{k+1}^2 - \sum_{j=1}^{k+1} \sigma_j^2 | \mathcal{A}_k\} &= E\{(S_k + X_{k+1})^2 - \sum_{j=1}^{k+1} \sigma_j^2 | \mathcal{A}_k\} \\
&= E\{S_k^2 + 2S_k X_{k+1} + X_{k+1}^2 | \mathcal{A}_k\} - \sum_{j=1}^{k+1} \sigma_j^2 \\
&= S_k^2 + 2S_k E(X_{k+1} | \mathcal{A}_k) + E(X_{k+1}^2) - \sum_{j=1}^{k+1} \sigma_j^2 \\
&= S_k^2 + 2S_k \cdot 0 + \sigma_{k+1}^2 - \sum_{j=1}^k \sigma_j^2 - \sigma_{k+1}^2 \\
&= S_k^2 - \sum_{j=1}^k \sigma_j^2 \quad \text{a.s.}
\end{aligned}$$

Thus  $\{S_k^2 - \sum_{j=1}^k \sigma_j^2, \mathcal{A}_k\}$  is a martingale.

6. Exercise 8.10.1, page 189: Show that  $\{T_k, \mathcal{A}_k\}_{n \leq k \leq N}$  is a martingale and that  $Var(T_N)$  is equal to the right side of (b) on page 188. (Exercise 8.11.1, page 249, PfS, 2000.)

**Solution:** If  $\{S_k, \mathcal{A}_k\}_{0 \leq k \leq N}$  is a 0-mean martingale and  $X_k \equiv S_k - S_{k-1}$  has  $Var(X_k) = \sigma_k^2 < \infty$  for all  $1 \leq k \leq N$ , then with

$$T_k \equiv S_n/b_n + \sum_{j=n+1}^k X_j/b_j, \quad \text{for } n \leq k \leq N,$$

$\{T_k, \mathcal{A}_k\}_{n \leq k \leq N}$  is a martingale. To see this, note that

$$\begin{aligned} E\{T_{k+1}|\mathcal{A}_k\} &= E\{S_n/b_n + \sum_{j=n+1}^{k+1} (X_j/b_j)|\mathcal{A}_k\} \\ &= S_n/b_n + \sum_{j=n+1}^k (X_j/b_j) + E\{(X_{k+1}/b_{k+1})|\mathcal{A}_k\} \\ &= T_k \quad \text{a.s.} \end{aligned}$$

since  $E(X_{k+1}|\mathcal{A}_k) = E(S_{k+1}|\mathcal{A}_k) - S_k = 0$  almost surely. Furthermore,

$$\text{Var}(T_N) = b_n^{-2} \sum_{k=1}^n \sigma_k^2 + \sum_{j=n+1}^N (\sigma_j^2/b_j^2)$$

since

$$\begin{aligned} \text{Var}(T_N) &= \text{Var}(S_n/b_n) + 2b_n^{-1} \text{Cov}(S_n, \sum_{j=n+1}^N (X_j/b_j)) + \text{Var}(\sum_{j=n+1}^N X_j/b_j) \\ &= b_n^{-2} \text{Var}(S_n) + 0 + \text{Var}(\sum_{j=n+1}^N X_j/b_j) \\ &= b_n^{-2} \sum_{k=1}^n \sigma_k^2 + \sum_{j=n+1}^N \sigma_j^2/b_j^2 \end{aligned}$$

via the following additional calculations:

$$\begin{aligned} \text{Var}(S_n) &= ES_n^2 = EE\{(S_{n-1} + X_n)^2|\mathcal{A}_{n-1}\} \\ &= E\{S_{n-1}^2 + 0 + E(X_n^2|\mathcal{A}_{n-1})\} \\ &= \text{Var}(S_{n-1}) + \text{Var}(X_n) = \text{Var}(S_{n-1}) + \sigma_n^2 \\ &= \dots = \sum_{j=1}^n \sigma_j^2, \end{aligned}$$

while, similarly,

$$\begin{aligned} \text{Var}\left(\sum_{j=n+1}^N X_j/b_j\right) &= E\left\{\left(\sum_{n+1}^N X_j/b_j\right)^2\right\} \\ &= EE\left\{\left(\sum_{n+1}^{N-1} X_j/b_j\right)^2 + 2b_N^{-1}X_N\left(\sum_{n+1}^{N-1} X_j/b_j\right) + b_N^{-2}X_N^2|\mathcal{A}_N\right\} \\ &= \text{Var}\left(\sum_{j=n+1}^{N-1} X_j/b_j\right) + 0 + b_N^{-2}\sigma_N^2 \\ &= \cdots = \sum_{j=n+1}^N \sigma_j^2/b_j^2. \end{aligned}$$