Statistics 522, Midterm Exam Solution

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1. **Define** *three* of the following four terms:
   (a) The conditional expectation of a random variable X given a (sub-) sigma-field $\mathcal{D}$.
   (b) A martingale, sub-martingale, and super-martingale.
   (c) A stopping time T (relative to a filtration $\mathcal{A}_n$).
   (d) The compensator of a sub-martingale.
   (e) A Brownian motion process $\mathbb{S}$ on $[0, \infty)$.

   Solution: See PfS, Course Notes.

2. Give careful **statements** of *three* of the following five theorems or results:
   (a) The S-mg convergence theorem.
   (b) Doob's decomposition theorem for sub-martinagles.
   (c) The simple optional sampling theorem.
   (d) The step-wise smoothing property of conditional expectations.
   (e) The interpretation of conditional expectations in terms of an (orthogonal) projection onto $L_2(\Omega, \mathcal{G}, P)$ where $\mathcal{G} \subset \mathcal{A}$.

   Solution: See PfS, Course Notes.

3. Suppose that X and Y are random variables on the probability space $(\Omega, \mathcal{A}, P)$ with $X \in L_2(P)$ and $Y \in L_2(P)$ (so that $XY \in L_1(P)$), and suppose that $\mathcal{D}$ is a sub sigma-field of $\mathcal{A}$. Show that

   $$E\{XE(Y|\mathcal{D})\} = E\{E(X|\mathcal{D})Y\} = E\{E(X|\mathcal{D})E(Y|\mathcal{D})\}.$$

   (This can be rewritten as

   $$\langle X, E(Y|\mathcal{D})\rangle = \langle E(X|\mathcal{D}), Y\rangle = \langle E(X|\mathcal{D}), E(Y|\mathcal{D})\rangle,$$

   and thus is the "self-adjointness property of the conditional expectation operator.)

   Solution: By computing conditionally on $\mathcal{D}$ we can write

   $$\begin{aligned} E\{XE(Y|\mathcal{D})\} &= E\{E[XE(Y|\mathcal{D})|\mathcal{D}]\} \\ &= E\{E(Y|\mathcal{D})E[X|\mathcal{D}]\} \\ &= E\{E[YE(X|\mathcal{D})|\mathcal{D}]\} \\ &= E\{YE(X|\mathcal{D})\}. \end{aligned}$$

4. Suppose that $X \in L_2(P)$ and $\mathcal{D}$ is a sub-sigma field. The conditional variance of X given $\mathcal{D}$ is defined by

$$Var(X|\mathcal{D}) = E\left\{(X - E(X|\mathcal{D}))^2|\mathcal{D}\right\}.$$

(a) Prove that

$$Var(X) = E[Var(X|\mathcal{D})] + Var(E(X|\mathcal{D})).$$

(b) Show that $E(X-Z)^2$ is minimized over all $\mathcal{D}$-measurable functions Z by $E(X|\mathcal{D})$.
(c) Interpret the formula in (a) geometrically.

Solution: (a) By adding and subtracting $E(X|\mathcal{D})$ we get

$$\begin{aligned}
Var(X) &= E(X - E(X|\mathcal{D}) + E(X|\mathcal{D}) - E(X))^2 \\
&= E(X - E(X|\mathcal{D}))^2 + E\{(E(X|\mathcal{D}) - E(X))^2\} \\
&\qquad + 2E\{(X - E(X|\mathcal{D}))(E(X|\mathcal{D}) - E(X))\} \\
&= E\{E[(X - E(X|\mathcal{D}))^2|\mathcal{D}]\} + Var[E(X|\mathcal{D})] \\
&= E\{Var[X|\mathcal{D}]\} + Var[E(X|\mathcal{D})]
\end{aligned}$$

since

$$\begin{aligned}
&E\{(X - E(X|\mathcal{D}))(E(X|\mathcal{D}) - E(X))\} \\
&\quad = E\{E[(X - E(X|\mathcal{D}))(E(X|\mathcal{D}) - E(X))|\mathcal{D}]\} \\
&\quad = E\{(E(X|\mathcal{D}) - E(X))E[X - E(X|\mathcal{D})|\mathcal{D}]\} \\
&\quad = E\{(E(X|\mathcal{D}) - E(X))(E[X|\mathcal{D}] - E[X|\mathcal{D}])\} \\
&\quad = E\{(E(X|\mathcal{D}) - E(X)) \cdot 0\} = 0.
\end{aligned}$$

(b) To show that $E(X-Z)^2$ is minimized over all $\mathcal{D}$-measurable functions Z by $E(X|\mathcal{D})$, write

$$\begin{aligned}
E(X-Z)^2 &= E(X - E(X|\mathcal{D}) + E(X|\mathcal{D}) - Z)^2 \\
&= E[(X - E(X|\mathcal{D}))^2] + E[(E(X|\mathcal{D}) - Z)^2] \\
&\qquad + 2E[(X - E(X|\mathcal{D}))(E(X|\mathcal{D}) - Z)] \\
&= E[(X - E(X|\mathcal{D}))^2] + E[(E(X|\mathcal{D}) - Z)^2] \\
&\geq E[(X - E(X|\mathcal{D}))^2]
\end{aligned}$$

with equality in the last inequality if and only if $Z = E(X|\mathcal{D})$. Here the third equality holds since

$$\begin{aligned}
&E[(X - E(X|\mathcal{D}))(E(X|\mathcal{D}) - Z)] \\
&\quad = E\{E[(X - E(X|\mathcal{D}))(E(X|\mathcal{D}) - Z)|\mathcal{D}]\} \\
&\quad = E\{(E(X|\mathcal{D}) - Z)E[X - E(X|\mathcal{D})|\mathcal{D}]\} \\
&\quad = E\{(E(X|\mathcal{D}) - Z)(E[X|\mathcal{D}] - E(X|\mathcal{D}))\} \\
&\quad = E\{(E(X|\mathcal{D}) - Z) \cdot 0\} = 0.
\end{aligned}$$

(c) When $E(X)=0$ the identity of (a) becomes

$$E(X^2)=E[(X-E(X|\mathcal{D}))^2]+E\{E[X|\mathcal{D}]^2\}.$$

This can be seen geometrically as a "Pythagorean" relationship as follows:

5. Suppose that $X_0=1$, and let $X_n \sim \text{Uniform}(0, X_{n-1})$ for $n \ge 1$. Let $\mathcal{A}_n \equiv \sigma[X_0, X_1, \ldots, X_n]$ for $n=0,1,\ldots$.
(a) Show that with $Y_n \equiv 2^n X_n$, $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a martingale, and hence that $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a non-negative super-martingale.
(b) Apply the s-mg convergence theorem to the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$.
(c) There is no convergence theorem stated for a non-negative super-martingale in PfS, but based on what you know about the s-martingale convergence theorem and the reversed martingale convergence theorem, state a convergence theorem for non-negative supermartingales and apply it to $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$. What is the a.s. limit of X_n in the present case?
(d) Is there any connection between the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ and Kakutani's product martingales?
(e) Use (d) to determine whether or not the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is uniformly integrable. Does convergence hold in L_1?
(f) Compute $E(X_{n+1}^2|\mathcal{A}_n)$ and $E(Y_{n+1}^2|\mathcal{A}_n)$.
(g) Use the computation in (f) to find a martingale related to $\{X_n^2\}$, and use it to compute $E(X_n^2)$ and $E(Y_n^2)$. Are either $\{X_n\}$ or $\{Y_n\}$ square-integrable?

Solution: (a) Since $X_{n+1} \sim \text{Uniform}(0, X_n)$, $E(X_{n+1}|\mathcal{A}_n) = X_n/2 \le X_n$ a.s., and hence $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a non-negative supermartingale. Furthermore,

$$E(Y_{n+1}|\mathcal{A}_n) = 2^{n+1}E(X_{n+1}|\mathcal{A}_n) = 2^{n+1}X_n/2 = 2^n X_n = Y_n \qquad \text{a.s.}$$

so that $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is a non-negative martingale.
(b) Now $E(Y_n)=E(Y_0)=E(X_0)=1$, for all n, and $Y_n \ge 0$, so $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is an L_1-bounded martingale, and hence by the s-martingale convergence theorem, $Y_n \to_{a.s.} Y_\infty$.
(c) **Theorem.** If $\{X_n, \mathcal{A}_n\}$ is a non-negative super-martingale, then $X_n \to_{a.s.} X_\infty$.
Proof. $W_n \equiv -X_n \le 0$ is a sub-martingale with $EW_n^+ = 0$. Thus by the s-mg convergence theorem, $W_n = -X_n \to_{a.s.} -X_\infty \equiv W_\infty$. In the present case, $X_n = 2^{-n}Y_n \to_{a.s.} 0 \cdot Y_\infty = 0$.
(d) Yes. Since $X_{n+1} \sim \text{Uniform}(0, X_n) =_d X_n \cdot \text{Uniform}(0,1) \equiv X_n W_{n+1}$ where $W_{n+1} \sim \text{Uniform}(0,1)$, it follows that

$$X_n = \prod_{k=1}^{n} W_k$$

where $W_1, W_2, \ldots$ are independent Uniform$(0,1)$ rv's. Thus $Y_n = 2^n X_n = \prod_{k=1}^n (2W_k)$ where $2W_k \sim \text{Uniform}(0,2)$ are i.i.d. with $E(2W_k)=1$. Thus Y_n is just the Kakutani

product martingale formed from i.i.d. Uniform$(0,2)$ random variables.
(e) Since $a_k = E(2W_k)^{1/2} = \sqrt{2}\int_0^1 u^{1/2}du = \sqrt{2}(2/3) < 1$,

$$\prod_{k=1}^{n} a_k = \left(\sqrt{2}\frac{2}{3}\right)^n \to 0$$

as $n \to \infty$; i.e. $\prod_{k=1}^{\infty} a_k = 0$. Thus by Kakutani's theorem the martingale $\{Y_n, \mathcal{A}_n\}_{n=0}^{\infty}$ is *not uniformly integrable*, and L_1-convergence fails: $1 = E(Y_n) \not\to E(Y_\infty) = 0$.
(f) Note that

$$\begin{aligned} E(X_{n+1}^2|\mathcal{A}_n) &= Var(X_{n+1}|\mathcal{A}_n) + (E(X_{n+1}|\mathcal{A}_n))^2 \\ &= \frac{1}{12}X_n^2 + \frac{1}{4}X_n^2 = \frac{1}{3}X_n^2 \qquad \text{a.s.} \end{aligned}$$

and therefore

$$E(Y_{n+1}^2|\mathcal{A}_n) = E(2^{2(n+1)}X_{n+1}^2|\mathcal{A}_n) = 2^{2(n+1)}\frac{1}{3}X_n^2 = \frac{4}{3}Y_n^2 .$$

(g) By the calculation in (f),

$$E((3/4)^{n+1}Y_{n+1}^2|\mathcal{A}_n) = (3/4)^{n+1}\frac{4}{3}Y_n^2 = (3/4)^n Y_n^2 \qquad \text{a.s.} ,$$

and therefore $Z_n \equiv (3/4)^n Y_n^2$ is a martingale. Thus $E((3/4)^n Y_n^2) = E(Z_n) = E(Z_0) = 1$, and it follows that $E(Y_n^2) = (4/3)^n$, and $E(2^{2n}X_n^2) = (4/3)^n$, or $E(X_n^2) = (1/3)^n$. Thus $\{X_n\}$ is square-integrable, but $\{Y_n\}$ is not square-integrable.

6. Suppose that $\{Z_n\}_{n=0}^{\infty}$ is a sequence of random variables with

$$P(Z_{n+1} = j|Z_n = i) = e^{-i}\frac{i^j}{j!}, \qquad i, j \in \{0, 1, 2, \ldots\}$$

with the convention that $P(Z_{n+1} = 0|Z_n = 0) = 1$. Also assume that $P(Z_0 = k_0) = 1$ for a fixed (possibly large) integer $k_0 \geq 1$.
(a) Show that $\{Z_n, \mathcal{A}_n\}$ is a martingale with mean k_0 (with respect to the filtration $\{\mathcal{A}_n\}$ with $\mathcal{A}_n = \sigma\{Z_0, \ldots, Z_n\}$ for $n \geq 0$).
(b) Show that with $Y_n \equiv P(Z_{n+1} = 0|\mathcal{A}_n) = P(Z_{n+1} = 0|Z_n)$, the process $\{Y_n, \mathcal{A}_n\}_{n\geq 0}$ is a sub-martingale.
(c) In fact, use Jensen's inequality to show that $\{Y_n, \mathcal{A}_n\}_{n\geq 0}$ is an almost surely strictly increasing sub-martingale.
(d) Use the result of (c) to show that $Y_n \to_{a.s.} 1$ and hence that $Z_n \to_{a.s.} 0$.
(e) Find the predictable variation process $\langle Z\rangle_n$ associated with the submartingale $\{Z_n^2, \mathcal{A}_n\}_{n=0}^{\infty}$. Show that $\{Z_n^2 - \langle Z\rangle_n\}$ is a zero mean martingale, and use this to

compute $E(Z_n^2)$ and $Var(Z_n)$.
(f) Show that for all $\lambda > 0$

$$P(\max_{0\le n<\infty} Z_n \ge \lambda) \le \frac{k_0}{\lambda}\,.$$

Solution: (a) Now $(Z_{n+1}|Z_n) \sim \text{Poisson}(Z_n)$, so

$$E(Z_{n+1}|\mathcal{A}_n) = E(Z_{n+1}|Z_n) = Z_n \quad a.s.$$

and hence $\{Z_n, \mathcal{A}_n\}$ is a martingale. The mean is k_0 since

$$E(Z_{n+1}) = E(Z_n) = \cdots = E(Z_0) = k_0.$$

(b) We compute

$$Y_n = P(Z_{n+1} = 0|\mathcal{A}_n) = P(Z_{n+1} = 0|Z_n) = e^{-Z_n} \quad \text{a.s.}.$$

Since e^{-v} is convex, it follows that $\{Y_n, \mathcal{A}_n\}_{n\ge 0}$ is a sub-martingale.
(c) Since e^{-v} is strictly convex, it follows that

$$Y_{n+1} = P(Z_{n+2} = 0|\mathcal{A}_{n+1}) = \exp(-Z_{n+1})$$

satisfies

$$\begin{aligned} E(Y_{n+1}|\mathcal{A}_n) &= E\{\exp(-Z_{n+1})|\mathcal{A}_n\} \ge \exp(-E(Z_{n+1}|\mathcal{A}_n)) \quad \text{a.s.} \\ &= \exp(-Z_n) = Y_n \end{aligned}$$

where the inequality holds with equality if and only if $Z_{n+1} = E(Z_{n+1}|\mathcal{A}_n)$ almost surely. But since $(Z_{n+1}|Z_n) \sim \text{Poisson}(Z_n)$, this equality fails to hold a.s. and hence the inequality in the last display is strict almost surely. Thus Y_n is strictly increasing (and bounded above by 1).
(d) Since Y_n is almost surely strictly increasing and bounded above by 1, it follows that $Y_n = e^{-Z_n} \nearrow 1$ almost surely. This implies that $Z_n \searrow 0$ a.s. as $n \to \infty$.
(e) Now $\langle Z\rangle_n$ is given by

$$\begin{aligned} \langle Z\rangle_n &= \sum_{k=1}^{n} E\{(\Delta Z_k)r|\mathcal{A}_{k-1}\} + EZ_0^2 \\ &= \sum_{k=1}^{n} Var(Z_k|\mathcal{A}_{k-1}) + k_0^2 \\ &= \sum_{k=1}^{n} Z_{k-1} + k_0^2. \end{aligned}$$

Thus $\{Z_n^2 - \langle Z\rangle_n, \mathcal{A}_n\}$ is a zero-mean martingale, and this implies that

$$\begin{aligned} E(Z_n^2) &= E\langle Z\rangle_n = E\{\sum_{k=1}^n Z_{k-1}\} + k_0^2 \\ &= nk_0 + k_0^2, \end{aligned}$$

and this yields

$$Var(Z_n) = E(Z_n^2) - (EZ_n)^2 = nk_0^2 + k_0^2 - k_0^2 = nk_0.$$

(f) By Doob's maximal inequality

$$P(\max_{1\le k\le n} Z_k \ge \lambda) \le \frac{EZ_n^+}{\lambda} = \frac{EZ_n}{\lambda} = \frac{k_0}{\lambda}.$$

Letting $n \to \infty$ yields

$$P(\max_{1\le k<\infty} Z_k \ge \lambda) \le \frac{k_0}{\lambda}$$

by the monotone convergence theorem.

Remark: Note that $Z_{n+1} = \sum_{j=1}^{Z_n} X_{nj}$ where the $X_{n,j}$'s are all i.i.d. Poisson(1) with $m = EX_{1,1} = 1$. Thus this example is a special case of the branching process example 13.4.5, page 366, PfS when $k_0 = 1$, and with general $k_0 > 1$ it is the branching process example with basic martingale having mean k_0 rather than 1. Note that (c) and (d) of the problem give a different proof of Theorem 3.4.1(ii) in this special case.