**Statistics 522, Problem Set 8**

Wellner; 3/6/2013

**Reading:**

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| Shorack, PfS Course Notes | Sections 9.3-, pages 201-224 |
| (PfS 2000 | sections 13.1-7, pages 341-364) |
| Shorack, PfS Course Notes | Sections 10.0 - 10.4, pages 225-252 |
| (PfS 2000, | sections 14.0-14.3, pages 365-382). |

**Due:** Wednesday, March 13, 2013.

**Reminder:** Final exam, Wednesday, March 20.

1. PfS Course Notes, Exercise 9.3.5; PfS (2000), Exercise 13.1.4, page 371. Show that the real part of a characteristic function (or $Re\phi(\cdot)$) is itself a characteristic function.

2. PfS Course Notes, Exercise 9.3.6; PfS (2000), Exercise 13.1.5, page 371. Let $\phi$ be a chf. Show that $c^{-1}\int_0^c \phi(tu)du$ is a chf.

3. PfS Course Notes, Exercise 9.4.2; PfS (2000), Exercise 13.2.2, page 348.

4. (a) Let $X_1, X_2, \ldots$, be i.i.d. random variables and let $Z_n \equiv n^{-1/2}\sum_{i=1}^n X_i$. For another sequence of i.i.d. random variables $X_1', X_2', \ldots$, with each $X_i' \stackrel{d}{=} X_i$ and all $X_i'$'s independent of the $X_i$'s, let $X_i^s \equiv X_i - X_i^s$ and set $Z_n^s \equiv n^{-1/2}\sum_{i=1}^n X_i^s$. Note that nothing has been assumed about finiteness of moments of the $X_i$'s (or $X_i$'s). Prove or disprove the following statement: $\overline{Z}_n \to_d N(0,1)$ if and only if the symmetrized random variables $\overline{Z}_n^s \to_d N(0,2)$.
(b) Now suppose that $X_1, X_2, \ldots$ are i.i.d. as in part (a), and suppose that $Z_n \equiv Z_{n,a,b} \equiv n^{1/2}(\overline{X} - a)/b$ for some $a \in \mathbb{R}$ and $b > 0$. What can you say about $a$ and $b$ if it is known that $Z_n \to_d N(0,1)$?

5. PfS Course Notes, Exercise 10.1.1, page 226; PfS (2000), Exercise 14.1.1, page 366.
For each $n \geq 1$, let $X_{n1}, \ldots, X_{nn}$ be i.i.d. with finite mean $\mu$. Use

characteristic functions to show the WLLN result that $\overline{X}_n \to_p \mu$ as $n \to \infty$. Equivalently, show that

$$\overline{X}_n \to_d \delta_\mu \equiv \text{the degenerate distribution with mass 1 at } \mu.$$

6. **Optional bonus problem:** PfS, Exercise 10.1.4, page 227.
(a) Suppose the hypotheses of the classical CLT hold. Show that

$$M_n \equiv \frac{1}{\sqrt{n}} \max_{1 \le k \le n} |X_{nk} - \mu| \to_p 0.$$

(b) Suppose that hypotheses of the classical Poisson Limit Theorem (see Theorem 1.2 on page 227 of PfS) hold. Show that

$$M_n \equiv \max_{1 \le k \le n} |X_{nk}| \to_d \text{Bernoulli}(1 - e^{-\lambda}).$$