Statistics 522, Problem Set 7

Wellner; 2/27/2013

Reading:

Shorack, PfS Course Notes	Sections 9.1-9.2, pages 193-200;
Shorack, PfS Course Notes	Sections 10.1 - 10.3, pages 225-249;
Wellner	Chapter 11, Sections 1-3, pages 1-21.

Due: Wednesday, March 6, 2013.

1. Exercise 9.2.4, PfS Course Notes, page 199. (Exercise 11.8.4, page 293, PfS 2000).

2. Exercise 11.6.6, page 34, Wellner, Chapter 11, notes.

3. Exercise 9.1.6, PfS Course Notes, page 197. (Exercise 11.7.6, page 291, PfS 2000).

4. Suppose that $X_1, \ldots, X_m$ are i.i.d. with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2 < \infty$; suppose that $Y_1, \ldots, Y_n$ are i.i.d. and independent of the X_i's with $E(Y_1) = \nu$ and $Var(Y_1) = \tau^2 < \infty$.
(a) Use the classical CLT (Theorem 11.2.2, W Chapter 11) to show that $\sqrt{m}(\overline{X}_m - \mu)/\sigma \to_d N(0,1)$ and that $\sqrt{n}(\overline{Y}_n - \nu)/\tau \to_d N(0,1)$.
(b) Let $N = m + n$ and set
$$D_{m,n} \equiv \sqrt{\frac{mn}{N}} \left(\overline{Y}_n - \overline{X}_m - (\nu - \mu) \right).$$
Use (a) to show that if $\lambda_N \equiv m/N \to \lambda \in [0,1]$ then
$$D_{m,n} \to_d \sqrt{\lambda}\tau Z - \sqrt{1-\lambda}\sigma Z' \sim N(0, \lambda\tau^2 + (1-\lambda)\sigma^2)$$
where $Z, Z' \sim N(0,1)$ are independent.
(c) Let $S_{m,n}^2 \equiv \lambda_N S_Y^2 + (1 - \lambda_N) S_X^2$ where $S_X^2 \equiv m^{-1} \sum_{i=1}^m (X_i - \overline{X}_m)^2$ and $S_Y^2 \equiv n^{-1} \sum_{j=1}^n (Y_j - \overline{Y}_n)^2$. Show that if $\lambda_N \to \lambda \in [0,1]$ then $S_{m,n}^2 \to_p \lambda\tau^2 + (1-\lambda)\sigma^2$.
(d) Use (b) and (c) to show that if $\lambda_N \to \lambda \in [0,1]$, then $T_{m,n} \equiv D_{m,n}/S_{m,n} \to_d Z \sim N(0,1)$.
(e) Use the result of (d) and the Helly selection theorem to show that $T_{m,n} \to_d Z \sim N(0,1)$ whenever $m \to \infty$ and $n \to \infty$.

5. **Bonus problem 1:** Exercise 11.6.5, page 34, Wellner, Chapter 11, notes (or equivalently Exercise 9.1.5, PfS Course Notes, page 196).

6. **Bonus problem 2:** Suppose that $X_1, \ldots, X_m$ are i.i.d. F and $Y_1, \ldots, Y_n$ are i.i.d. G. Let $\mathbb{F}_m(x) \equiv m^{-1} \sum_{i=1}^m 1_{(-\infty,x]}(X_i)$ and $\mathbb{G}_n(x) \equiv n^{-1} \sum_{j=1}^n 1_{(-\infty,x]}(Y_j)$. To test the hypothesis $F = G$ consider the test statistic

$$D_{m,n} \equiv \sqrt{\frac{mn}{N}} \sup_{x \in \mathbb{R}} |\mathbb{F}_m(x) - \mathbb{G}_n(x)| \equiv \sqrt{\frac{mn}{N}} \|\mathbb{F}_m - \mathbb{G}_n\|.$$

Under the null hypothesis $F = G$ continuous, show that $D_{m,n} \rightarrow_d \|\mathbb{U}\| \equiv \sup_{0 \le t \le 1} |\mathbb{U}(t)|$ whenever $m \rightarrow \infty$ and $n \rightarrow \infty$ where $\mathbb{U}$ is a standard Brownian bridge process.