Statistics 522, Problem Set 5

Wellner; 2/13/2013

Reading:

Shorack, PfS Course Notes	Chapter 13, Sections 13.5 - 13.8, pages 369 -382.
Shorack, PfS (2000)	Chapter 18, Sections 18.5-18.8, pages 487 - 500.

Due: Wednesday, February 20, 2013.

Reminder: Midterm exam, Friday, February 15.

1. Polyá's urn: At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn, and is replaced together with a new ball of the same color. Just after time n, there are therefore $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n + 1)/(n + 2)$, the proportion of black balls in the urn just after time n. Prove that (relative to a natural filtration which you should specify) M_n is a martingale. Prove that $P(B_n = k) = 1/(n+1)$ for $0 \le k \le n$. What is the distribution of $\Theta \equiv \lim_n M_n$? Prove that for $0 < \theta < 1$,

$$N_n^\theta \equiv \frac{(n+1)!}{B_n!(n-B_n)!}\theta^{B_n}(1-\theta)^{n-B_n}$$

defines a martingale N_n^θ.

2. Let $X_1, X_2, \ldots$ be i.i.d rv's with $P(X = 1) = p$, $P(X = -1) = 1-p \equiv q$, where $0 < p < 1$ and $p \neq q$. Suppose that a, b are integers with $-a < 0 < b$. Define

$$S_n = X_1 + \ldots + X_n, \qquad T \equiv \inf\{n : S_n = -a, \text{or } \ S_n = b\}.$$

Let $\mathcal{F}_n \equiv \sigma[X_1, \ldots, X_n]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prove that $M_n \equiv (q/p)^{S_n}$ and $N_n = S_n - n(p - q)$ define martingales M_n and N_n. How would you use these martingales to deduce the values of $P(S_T = -a)$ and $E(S_T)$? [Hint: see PfS, Course Notes, pages 381-382.]

3. Exercise 13.7.2, PfS, Course Notes page 382. Suppose that S_μ is Brownian motion with drift: $S_\mu(t) = S(t) + \mu t$ for $t \ge 0$. Let $\tau_{ab} \equiv \tau \equiv \inf\{t \ge 0 : S_\mu(t) = -a \text{ or } b\}$ where $-a < 0 < b$.
   Claim 1: $S_0(t)$, $S_0^2(t) - t$, $S_\mu(t) - \mu t$ are mean 0 martingales, and, with $\theta = -2\mu$,

$$\exp(\theta[S_\mu(t) - \mu t] - \theta^2 t/2) = \exp(-2\mu[S(t) + \mu t])$$

   is a mean 1 martingale.
   Claim 2: If $\mu = 0$, $P(S(\tau) = -a) = b/(a+b)$ and $E\tau = ab$.
   Claim 3: If $\mu \neq 0$, then

$$P(S(\tau) = -a) = \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}}$$

   and

$$E(\tau) = \frac{b}{\mu} - \frac{a+b}{\mu} \frac{1 - e^{2\mu b}}{1 - e^{2\mu(a+b)}} .$$

   Claim 4: If $\mu < 0$, then $P(\|S_\mu\|_0^\infty \ge b) = \exp(-2|\mu| b)$ for all $b > 0$; i.e. $\|S_\mu\|_0^\infty \sim \text{Exponential}(2|\mu|)$.
   (Note the analogies with problem 2.)