Statistics 522, Problem Set 4

Wellner; 2/6/2013

Reading:	Shorack, PfS (Course Notes), Chapter 13, sections 18.4 - 18.7, pages 363 - 382
Due:	Wednesday, February 13, 2013.
Reminder:	Makeup lecture 2: 8 February (Friday) from 12:30 - 1:20, Denny 315.
Midterm Exam:	Friday, February 15.

1. Exercise 13.3.6, PfS Course Notes, page 359. [Exercise 18.3.5, PfS (2000), page 477.] Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a sub-martingale with $X_n \geq 0$. Let $r > 1$. Then $\{X_n^r\}$ is uniformly integrable if and only if $\{X_n^r\}$ is integrable.

2. Exercise 13.3.7, PfS Course Notes, page 359. [Exercise 18.3.7, PfS (2000), page 477.] Let $r > 1$. Let $\{X_n, \mathcal{A}_n\}_{n=0}^{\infty}$ be a martingale. Then the following are equivalent:
   (10) The $|X_n|^r-$process is integrable.
   (11) $X_n \to_r X_\infty$
   (12) The X_n's are uniformly integrable (thus $X_n \to$ (some X_∞) a.s.) and $X_\infty \in L_r$.
   (13) The $|X_n|^r$'s are uniformly integrable.
   (14) $\{|X_n|^r, \mathcal{A}_n\}_{n=0}^{\overline{\infty}}$ is a submg and $E|X_n|^r \nearrow E|X_\infty|^r$.

3. Exercise 13.4.3, PfS Course Notes, page 365. (Conditional Borel-Cantelli) Let $\mathcal{A}_n$ be an increasing sequence of $\sigma-$fields in $\mathcal{A}$, and let $A_n \in \mathcal{A}_n$. Show that $[A_n \ i.o.] = [\omega : \sum_{n=1}^{\infty} P(A_n|\mathcal{A}_{n-1}) = \infty]$ almost surely.

4. Exercise 13.4.4, PfS Course Notes, page 366. [Exercise 18.4.3, PfS (2000), page 484.]

5. Suppose that $X_1, X_2, \ldots$ are independent random variables on $(\Omega, \mathcal{A})$ and that X_n has density p_n or q_n under P or Q respectively where p_n and q_n are (for simplicity) everywhere positive on $\mathbb{R}$. Let $\mathcal{F} = \sigma[X_1, X_2, \ldots]$ and $\mathcal{F}_n = \sigma[X_1, \ldots, X_n]$ for $n \geq 1$. Let $Y_n \equiv q_n(X_n)/p_n(X_n)$.
   (a) Show that
   $$M_n \equiv \frac{dQ}{dP}\Big|_{\mathcal{F}_n} = Y_1 \cdots Y_n$$
   where the Y_n's are independent and have mean 1 under P; Hence the likelihood ratio martingale of Example 1.14 is the Kakutani product martingale of Example 1.15.
   (b) Show that Q is absolutely continuous relative to P on $\mathcal{F}$ if and only if the martingale $\{M_n, \mathcal{F}_n\}$ is uniformly integrable.

(c) Conclude from Kakutani's theorem (PfS Example 4.4, pages 482-483) that $Q \ll P$ on $\mathcal{F}$ if and only if

$$\prod_{n=1}^{\infty} E(Y_n^{1/2}) = \prod_{n=1}^{\infty} \int_{\mathbb{R}} \sqrt{p_n(x)q_n(x)}dx > 0\,.$$

(d) Construct two examples of sequences p_n and q_n, one in which the condition in (c) holds and one in which it fails. What is the statistical meaning when it holds and when it fails?