Statistics 522, Problem Set 2

Wellner; 1/16/2013

Reading:		
Shorack,	PfS Course Notes,	Chapter 5, Section 4, pages 98 - 101;
Shorack,	PfS Course Notes,	Chapter 12, Sections 1-3, pages 301 - 315.
Due:	Wednesday,	January 23, 2013.
Reminder:	Jon gone	29 January (Wednesday) and 1 February (Friday)
	Makeup lectures:	25 January (Friday) and
		8 February (Friday) from 12:30 - 1:20.

1. Show that $Z = E(Y|\mathcal{D})$ minimizes $E(Y-Z)^2$ among all $\mathcal{D}-$measurable random variables $Z \in \mathcal{H}_{\mathcal{D}}$.

2. Suppose that Y is a random variable defined on $(\Omega, \mathcal{A}, P)$ and that $EY^2 < \infty$. Moreover, suppose $\mathcal{D} \subset \mathcal{A}$ is a sub-$\sigma-$field of $\mathcal{A}$.
(a) Show that $Var(Y) = Var(E(Y|\mathcal{D}) + E\{Var(Y|\mathcal{D})\}$.
(b) Can you relate this to the first problem above?

3. PfS, Exercise 7.5.3, PfS Course Notes, page 145:
Theorem 5.4. Suppose that $X : (\Omega, \mathcal{A}, P) \to (M_1, \mathcal{G}_1)$ and $Y : (\Omega, \mathcal{A}, P) \to (M_2, \mathcal{G}_2)$ where $(M_1, \mathcal{G}_1)$ and $(M_2, \mathcal{G}_2)$ are Borel spaces. Then $(X, Y) : (\Omega, \mathcal{A}, P) \to (M_1 \times M_2, \mathcal{G}_1 \times \mathcal{G}_2)$. Furthermore

$$\text{a regular conditional probability} \quad P(A|X = x) \text{ exists}$$

for sets $A \in \tilde{\mathcal{A}} \equiv Y^{-1}(\mathcal{G}_2) \subset \mathcal{A}$ and for $x \in M_1$. Let $E|h(X, Y)| < \infty$.
(a) Then

$$E\{h(X, Y)|X = x\} = \int_{M_2} h(x, y) dP(y|X = x) \quad \text{a.s.}$$

(b) If X and Y are independent, then

$$E\{h(X, Y)|X = x\} = E\{h(x, Y)\} \quad \text{a.s.}$$

4. Suppose that $X, Y \in L_1(\Omega, \mathcal{F}, P)$ and that $E(Y|X) = X$ a.s. and $E(X|Y) = Y$ a.s. Prove that $P(X = Y) = 1$. (See e.g. Exercise 9.2, Williams, *Probability with Martingales*, page 231.)

5. (Symmetry and conditional expectation). Let $X_1, X_2, \ldots$ be i.i.d. random variables with the same distribution as X where $E|X| < \infty$. Let $S_n \equiv X_1 + \cdots + X_n$, and define

$$\mathcal{G}_n \equiv \sigma\left[S_n, S_{n+1}, \ldots\right] = \sigma\left[S_n, X_{n+1}, X_{n+2}, \ldots\right].$$

Show that $E(X_1|\mathcal{G}_n) = E(X_1|S_n) = n^{-1}S_n$ almost surely. [Hint: Note that $\sigma\left[X_{n+1}, X_{n+2}, \ldots\right]$ is independent of $\sigma\left[X_1, S_n\right]$, and use symmetry to show that $E(1_{[S_n \in B]}X_1) = E(1_{[S_n \in B]}X_2) = \cdots = E(1_{[S_n \in B]}X_n)$.]