Statistics 522, Problem Set 8 Solutions

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1. PfS, Exercise 13.1.4, page 371.
   Show that the real part of a characteristic function (or $Re\phi(\cdot)$) is itself a characteristic function.

   Solution: First solution: Let $\phi = \phi_X$ be the characteristic function of the random variable X with distribution function F on $\mathbb{R}$, and let ϵ be a Rademacher random variable independent of X. Then ϵX has characteristic function

$$
\begin{aligned}
\phi_{\epsilon X}(t) &= Ee^{it\epsilon X} = E_X\{E_\epsilon(e^{it\epsilon X})\} \\
&= E_X\left\{\frac{1}{2}e^{itX} + \frac{1}{2}e^{-itX}\right\} \\
&= E\{\cos(tX)\} = \mathrm{Re}(\phi(t)).
\end{aligned}
$$

   Thus $\mathrm{Re}(\phi)$ is the characteristic function of ϵX where ϵ is a Rademacher random variable independent of X.

   Second solution: Let $\phi = \phi_X$ be the characteristic function of the random variable X with distribution function F on $\mathbb{R}$. Then the claim is that $\mathrm{Re}(\phi_X) = E\cos(tX) = \int \cos(tx)dF_X(x)$ is a characteristic function. Consider the distribution function G defined by

$$
G(y) \equiv \left[F_X(y) + 1 - F_X(-y-)\right]/2 = (P(X \le y) + P(-X \le y))/2.
$$

   Then G is symmetric:

$$
\begin{aligned}
G(-y) &= (P(X \le -y) + P(-X \le -y))/2 \\
&= (P(-X \ge y) + P(X \ge y))/2 = 1 - G(y-) \equiv 1 - G_-(y)
\end{aligned}
$$

   for all $y \in \mathbb{R}$. Let Y have distribution function G on $\mathbb{R}$; thus $-Y \stackrel{d}{=} Y$. By Proposition 13.1.1 (e), the characteristic function of Y is real-valued.

Thus we have

$$\begin{aligned}
\phi_Y(t) &= Ee^{itY} = E\cos(tY) = \int \cos(ty)dG(y) \\
&= \frac{1}{2}\int \cos(ty)d\{F_X(y) + (1 - F_X(-y-))\} \\
&= \frac{1}{2}\int \cos(ty)dF_X(y) + \frac{1}{2}\int \cos(ty)dF_X(-y-) \\
&= \frac{1}{2}\int \cos(ty)dF_X(y) + \frac{1}{2}\int \cos(-ty)dF_X(y-) \\
&= \frac{1}{2}\int \cos(ty)dF_X(y) + \frac{1}{2}\int \cos(ty)dF_X(y) \\
&= \int \cos(ty)dF_X(y)
\end{aligned}$$

since $\cos(-v) = \cos(v)$ for all $v \in \mathbb{R}$ and since $F_X(\cdot -)$ and F_X determine the same measure. Thus $\mathrm{Re}(\phi_X)$ is the characteristic function of the random variable Y with distribution function G.

To connect the two solutions, note that $Y \equiv \epsilon X$ has distribution function G given by

$$G(y) = P(\epsilon X \le y) = E\{P(\epsilon X \le y|\epsilon)\}$$

where

$$P(\epsilon X \le y|\epsilon) = \begin{cases} P(X \le y) = F(y) & \text{on} \quad [\epsilon = 1] \\ P(-X \le y) = P(X \ge -y) = 1 - F(-y-) & \text{on} \quad [\epsilon = -1]\,. \end{cases}$$

Thus

$$G(y) = \frac{1}{2}\{F(y) + (1 - F(-y-))\}\,.$$

2. PfS, Exercise 13.4.4, page 381.
Verify that if g and g' are in $\mathcal{L}_1(R, \mathcal{B}, \lambda)$, then $|g(x)| \to 0$ as $x \to \infty$.

Solution: If $g' \in \mathcal{L}_1(\mathbb{R}, \mathcal{B}, \lambda)$, then the function h defined by

$$h(x) = \int_{-\infty}^{x} g'(y)dy$$

is absolutely continuous, $h'(x) = g'(x)$ a.e. λ, and $h(x) = g(x) + c$ for some c. Since

$$|h(x)| \le \int_{-\infty}^{x} |g'(y)| dy \to 0 \qquad \text{as } \ x \to -\infty$$

by the dominated convergence theorem. Thus

$$0 = \lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} (g(x) + c) = g(-\infty) + c,$$

so $c = -g(-\infty)$ and we have

$$g(x) = g(-\infty) + \int_{-\infty}^{x} g'(y) dy.$$

Suppose $g(-\infty) \neq 0$. Then, since g is integrable

$$\begin{aligned} \infty &> \int |g(x)| dx = \int \left| g(-\infty) + \int_{-\infty}^{x} g'(y) dy \right| dx \\ &\ge \int_{-\infty}^{x} \frac{1}{2} |g(-\infty)| dy = \infty \end{aligned}$$

by choosing x_0 so small that $| \int_{-\infty}^{x} g'(y) dy | le |g(-\infty)|/2$ for all $x < x_0$. (This follows from the triangle inequality since

$$\begin{aligned} |g(-\infty)| &= \left| g(-\infty) + \int_{-\infty}^{x} g'(y) dy + \int_{-\infty}^{x} g'(y) dy \right| \\ &\le \left| g(-\infty) + \int_{-\infty}^{x} g'(y) dy \right| + \left| \int_{-\infty}^{x} g'(y) dy \right| \\ &\le \left| g(-\infty) + \int_{-\infty}^{x} g'(y) dy \right| + |g(-\infty)|/2 \end{aligned}$$

for $x < x_0$.) But this is a contradiction, and hence $g(-\infty) = 0$. A similar argument works for $x \to \infty$ by setting $h(x) \equiv \int_{x}^{\infty} g'(y) dy$.

3. PfS, Exercise 14.1.1, page 392.
   For each $n \ge 1$, let $X_{n1}, \ldots, X_{nn}$ be i.i.d. with finite mean μ. Use characteristic functions to show the WLLN result that $\overline{X}_n \to_p \mu$ as $n \to \infty$. Equivalently, show that

$$\overline{X}_n \to_d \delta_\mu \equiv \text{the degenerate distribution with mass 1 at } \mu.$$

Solution: Let $\mu \equiv E(X_1)$ and $\phi(t) \equiv E\exp(itX_1)$. Since $E|X_1| < \infty$, it follows from inequality 4.2 that

$$|\phi(t) - 1 - it\mu|/|t| \to 0 \qquad \text{as } \ t \to 0\,.$$

Therefore we have

$$\begin{aligned}\phi_{\overline{X}_n}(t) &= \phi(t/n)^n = \left\{1 + \frac{it\mu}{n} + \frac{\phi(t/n) - 1 - i(t/n)\mu}{t/n}\frac{t}{n}\right\}^n \\ &= \left\{1 + \frac{it}{n}\left(\mu + \frac{\phi(t/n) - 1 - i(t/n)\mu}{it/n}\right)\right\}^n \\ &\to \exp(it\mu)\,,\end{aligned}$$

the characteristic function of μ. Hence by the Cramér - Lévy continuity theorem, $\overline{X}_n \to_d \mu$. But since convergence in distribution to a constant implies convergence in probability (to the same constant), it follows that $\overline{X}_n \to_p \mu$.

4. PfS, Exercise 14.1.4, page 393.
(a) Suppose the hypotheses of the classical CLT hold. Show that

$$M_n \equiv \frac{1}{\sqrt{n}} \max_{1 \le k \le n} |X_{nk} - \mu| \to_p 0.$$

(b) Suppose that hypotheses of the classical Poisson Limit Theorem (see Theorem 1.2 on page 393 of PfS) hold. Show that

$$M_n \equiv \max_{1 \le k \le n} |X_{nk}| \to_d \text{Bernoulli}(1 - e^{-\lambda}).$$

Solution: (a) In this case the X_{nk}'s are i.i.d. with finite mean μ and variance $\sigma^2 < \infty$. By theorem 10.4.3, with $r = 2$, it follows that

$$M_n^2 = \frac{1}{n} \max_{1 \le k \le n} |X_k - \mu|^2 \to_p 0$$

if and only if

$$x^2 P(|X_1| > x) \to 0 \qquad \text{as } \ x \to \infty. \tag{1}$$

But $Var(X_1) = \sigma^2 < \infty$ implies that (1) holds. Thus $M_n \to_p 0$ and $M_n \to_p 0$ by the continuous mapping theorem.
(b) If the X_{nk}'s are independent Bernoulli (λ_{nk}) with

$$\lambda_n \equiv \sum_{k=1}^{n} \lambda_{nk} \to \lambda, \tag{2}$$

$$\max_{1 \le k \le n} \lambda_{nk} \to 0, \tag{3}$$

then by Lemma 13.4.3, PfS page 379,

$$\begin{aligned} P(M_n \le x) &= P(\max_{1 \le k \le n} X_{nk} \le x) \\ &= \begin{cases} 0, & x < 0, \\ \prod_{k=1}^{n}(1 - \lambda_{nk}), & 0 \le x < 1, \\ 1, & x \ge 1, \end{cases} \\ &\to \begin{cases} 0, & x < 0, \\ e^{-\lambda}, & 0 \le x < 1, \\ 1, & x \ge 1 \end{cases} \\ &\equiv G(x) \end{aligned} \tag{4}$$

where the first two hypotheses of Lemma 13.4.3 hold by virtue of (2) and (3) and the third condition of the lemma holds since

$$M_n \equiv \sum_{k=1}^{n} |\lambda_{nk}| = \sum_{k=1}^{n} \lambda_{nk} \equiv \lambda_n$$

satisfies $\max_{1 \le k \le n} \lambda_{nk} M_n \to 0$. Now note that the distribution function G in (4) is exactly that of a Bernoulli$(1 - e^{-\lambda})$ random variable.

5. PfS, Exercise 14.2.9, page 403. The following are equivalent:
(a) $|X_{nk}|$'s are uan, meaning that $\max_{1 \le k \le n} P(|X_{nk}| \ge \epsilon) \to 0$ for all $\epsilon > 0$.
(b) $\max_{1 \le k \le n} |\phi_{nk}(t) - 1| \to 0$ uniformly on every finite interval of t's.
(c) $\max_{1 \le k \le n} \int \alpha dF_{nk}(x) \to 0$ for $\alpha(x) \equiv x^2 \wedge 1$.

Solution: First, (a) implies (b): Fix $\delta > 0$ and $T > 0$. Choose $\epsilon \le \delta/T$. Note that

$$\phi_{nk}(t) = Ee^{itX_{nk}} = E(e^{itX_{nk}} 1_{[|X_{nk}| \le \epsilon]}) + E(e^{itX_{nk}} 1_{[|X_{nk}| > \epsilon]}) \,.$$

Moreover, from the proof of Lemma 4.2,

$$\sup_{|t|\le T} |(e^{itx}-1)1_{[|x|\le\epsilon]}| \le \sup_{|t|\le T} |tx| 1_{[|x|\le\epsilon]} \le T\epsilon\,.$$

It follows that

$$\begin{aligned}
\max_{1\le k\le n}\sup_{|t|\le T}|\phi_{nk}(t)-1| &\le \max_{1\le k\le n}\sup_{|t|\le T}|E(e^{itX_{nk}}1_{[|X_{nk}|\le\epsilon]}-1)| \\
&\qquad + 2\max_{1\le k\le n} P(|X_{nk}|>\epsilon) \\
&\le \max_{1\le k\le n}\sup_{|t|\le T}|E[(e^{itX_{nk}}-1)1_{[|X_{nk}|\le\epsilon]}]| \\
&\qquad + 3\max_{1\le k\le n} P(|X_{nk}|>\epsilon) \\
&\le \epsilon T + 3\max_{1\le k\le n} P(|X_{nk}|>\epsilon) \\
&\to \epsilon T = \delta
\end{aligned}$$

as $n\to\infty$. But $\delta>0$ was arbitrary, so (b) holds.

Now (b) implies (a): an inequality in the same spirit as the one we used to prove the continuity theorem, Inequality 13.3.1, page 293, is as follows:

$$P(|X|\ge\epsilon) \le \frac{\epsilon}{2}\int_{[|t|\le 2/\epsilon]} |1-\phi(t)|\,dt\,. \tag{5}$$

We will prove this below. Suppose that (b) holds. Note that (5) implies that

$$\begin{aligned}
\max_{k\le n} P(|X_{nk}|\ge\epsilon) &\le \frac{\epsilon}{2}\max_{k\le n}\int_{[|t|\le 2/\epsilon]}|1-\phi_{nk}(t)|\,dt \\
&\le 2\max_{k\le n}\sup_{|t|\le 2/\epsilon}|1-\phi_{nk}(t)| \\
&\to 0
\end{aligned}$$

and hence (b) implies (a). To see that (5) holds, note that for $T\in(0,\infty)$ we have, by Fubini's theorem,

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T}\phi(t)dt &= \frac{1}{2T}\int_{-T}^{T} E(\cos(tX)+i\sin(tX))\,dt \\
&= \frac{1}{2T}E\left\{\int_{-T}^{T}(\cos(tX)+i\sin(tX))\,dt\right\} \\
&= E\left(\frac{\sin(TX)}{TX}\right)\,.
\end{aligned}$$

It follows that

$$
\begin{aligned}
\Big|\frac{1}{2T}\int_{-T}^{T}\phi(t)dt\Big| &\le E\Big|\frac{\sin(TX)}{TX}\Big| \\
&\le E\Big|\frac{\sin(TX)}{TX}\Big|1_{[|X|\ge\epsilon]} + E\Big|\frac{\sin(TX)}{TX}\Big|1_{[|X|<\epsilon]} \\
&\le \frac{1}{T\epsilon}P(|X|\ge\epsilon) + 1 - P(|X|\ge\epsilon)
\end{aligned}
$$

since $|\sin(y)| \le 1$ and $|\sin(y)/y| \le 1$. Choosing $T = 2/\epsilon$ yields

$$
\Big|\frac{\epsilon}{4}\int_{-2/\epsilon}^{2/\epsilon}\phi(t)dt\Big| \le 1 - \frac{1}{2}P(|X|\ge\epsilon)
$$

or, equivalently,

$$
\begin{aligned}
P(|X|\ge\epsilon) &\le 2 - \Big|\frac{\epsilon}{2}\int_{-2/\epsilon}^{2/\epsilon}\phi(t)dt\Big| \\
&= \frac{\epsilon}{2}\int_{|t|\le 2/\epsilon} dt - \Big|\frac{\epsilon}{2}\int_{-2/\epsilon}^{2/\epsilon}\phi(t)dt\Big| \\
&\le \frac{\epsilon}{2}\int_{-2/\epsilon}^{2/\epsilon}|1-\phi(t)|dt\,;
\end{aligned}
$$

i.e. (5) holds.

Note that (c) implies (a) easily since, for $\epsilon \in (0, 1]$,

$$
1_{[|x|\ge\epsilon]} \le \frac{x^2}{\epsilon^2}\wedge 1 \le \frac{x^2\wedge 1}{\epsilon^2} = \frac{\alpha(x)}{\epsilon^2}\,,
$$

and hence

$$
P(|X_{nk}|\ge\epsilon) \le \epsilon^{-2}E\alpha(X_{nk})\,.
$$

Finally, (a) implies (c): for any $\epsilon < 1$,

$$
\begin{aligned}
E\alpha(X_{nk}) &= E\alpha(X_{nk})1_{[|X_{nk}|\le\epsilon]} + E\alpha(X_{nk})1_{[|X_{nk}|>\epsilon]} \\
&\le \epsilon^2 + P(|X_{nk}|\ge\epsilon)\,,
\end{aligned}
$$

and hence

$$
\max_{k\le n} E\alpha(X_{nk}) \le \epsilon^2 + \max_{k\le n} P(|X_{nk}|\ge\epsilon) \to \epsilon^2\,.
$$

Since this holds for arbitrary $\epsilon > 0$, (c) holds.

6. **Optional bonus problem:** PfS, Exercise 14.2.10, page 403. Let $\overline{Z}_n$ denote an appropriately normalized $\overline{X}_n$ for a single i.i.d sequence of rv's. Prove or disprove the following statement: $\overline{Z}_n \to_d N(0,1)$ if and only if the symmetrized random variables $\overline{Z}_n^s \to_d N(0,2)$. (Recall Definition 10.3.1, PfS, page 228.)

7. **Optional bonus problem:** Suppose that $X_1, X_2, \ldots$ are i.i.d. with characteristic function $\phi \equiv \phi_X$, and let $N \sim \text{Poisson}(\lambda)$ be independent of all the X_i's. Show that $S \equiv X_1 + \cdots + X_N$ has characteristic function ϕ_S given by

$$\phi_S(t) = \exp(\lambda(\phi_X(t) - 1)).$$

(See PfS, Example 15.1.1, page 430.)