Statistics 522, Problem Set 7 Solutions

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1. Graph the first few g_{nj}'s and h_{nj}'s introduced in the handout on the Haar function construction of Brownian bridge and Brownian motion.

Solution:

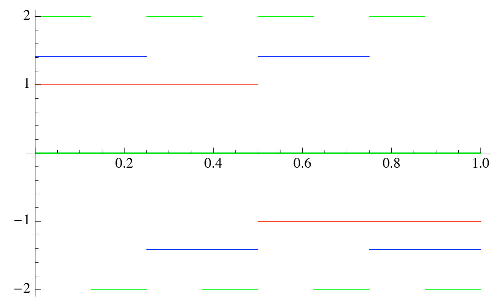

Figure 1: The Haar functions for $n = 0, 1, 2$.

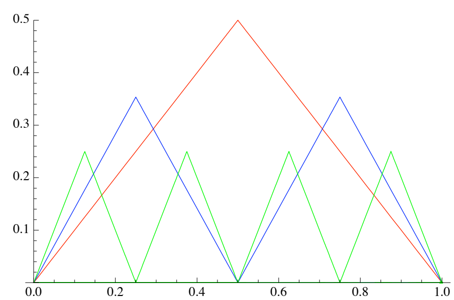

Figure 2: The Schauder functions for $n = 0, 1, 2$.

2. Justify the interchange of expectation and summation used in the proof of the Theorem in the handout.

Solution: Note that $\mathbb{U}(t) = \sum_{n=0}^{\infty} \mathbb{V}_n(t)$ where

$$\mathbb{V}_n(t) = \sum -j = 0^{2^n - 1} h_{nj}(t) X_{nj}$$

satisfies (from page 2, line 4)

$$|\mathbb{V}_n(t)| \leq 2^{-n/2-1} \max_{0 \leq j \leq 2^n - 1} |X_{nj}| \equiv 2^{-n/2-1} Z_n$$

Therefore

$$\begin{aligned} |\mathbb{U}(t)| &\leq \sum_{n=0}^{\infty} |\mathbb{V}_n(t)| \leq \sum_{n=0}^{\infty} 2^{-n/2-1} \max_{0 \leq j \leq n} |X_{nj}| \\ &= \sum_{n=0}^{\infty} 2^{-n/2-1} Z_n. \end{aligned}$$

Now the X_{nj}'s are i.i.d. $N(0,1)$ with $E|X_{nj}|^r = E|X_{1,0}|^r < \infty$ for every $r > 0$. Thus $Z_n^r / 2^n \to_{a.s.} 0$ and $E(Z_n^r/2^n) \to 0$. Therefore, by Liapunov's inequality,

$$E(Z_n/2^{n/r}) \leq \{E(Z_n^r/2^n)\}^{1/r} \to 0 \qquad \text{as } n \to \infty$$

for every $r > 0$. This implies that

$$\sup_{n \geq 1} E(Z_n/2^{n/r}) \leq M_r < \infty,$$

for each $r > 0$. Choosing $r = 4$ yields

$$\sup_{n \geq 1} EZ_n \leq M_4 2^{n/4}.$$

Thus by Tonelli's theorem

$$E|\mathbb{U}(t)| \leq \sum_{n=0}^{\infty} 2^{-n/2-1} EZ_n \leq M_4 \sum_{n=0}^{\infty} 2^{-n/2-1} 2^{n/4} = 2^{-1} M m_4 \sum_{n=0}^{\infty} 2^{-n/4} < \infty.$$

By Fubini's theorem, this justifies the interchange of expectation and summation needed to show that

$$E\mathbb{U}(t) = \sum_{n=0}^{\infty} E\mathbb{V}_n(t) = \sum_{n=0}^{\infty} \sum_{j=0}^{2^n - 1} h_{nj}(t) E X_{nj} = 0.$$

To show that the interchange on page 4, line 4, of the handout is valid, first note that the argument above shows that

$$\mathbb{U}(t)^2 = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} \mathbb{V}_n(t)\mathbb{V}_{n'}(t)$$

so that

$$\mathbb{U}(t)^2 \le \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} 2^{-n/2-1}2^{-n'/2-1} Z_n Z_{n'}$$

where

$$E(Z_n Z_n') = \left\{ \begin{array}{ll} E(Z_n)E(Z_{n'}) & \text{for } n \neq n' \\ E(Z_n^2) & \text{for } n = n' \end{array} \right\} \le \begin{array}{ll} M_4 2^{n/4} M_4 2^{n/4} & \text{for } n \neq n' \\ K_4 2^{n/2} & \text{for } n = n' \end{array}$$

since

$$\max_{n\ge 1} E(Z_n^2/2^{n/r}) \le \max_{n\ge 1} \left\{ E(Z_n^{2r}/2^n) \right\}^{1/r} \le K_r$$

and hence

$$\max_{n\ge 1} EZ_n^2 \le K_2 2^{n/2} \qquad \text{if } r = 2.$$

Therefore by Tonelli's theorem

$$\begin{aligned} E\mathbb{U}(t)^2 &\le \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} 2^{-n/2-1}2^{-n'/2-1}(M_4 \vee \sqrt{K_2})^2 \cdot 2^{n/4} \cdot 2^{n'/4} \\ &= (M_4 \vee \sqrt{K_2})^2 \left(\sum_{n=0}^{\infty} 2^{-n/4-1} \right)^2 < \infty \end{aligned}$$

and this justifies the interchange on page 4, line 4, via the Cauchy-Schwarz inequality.

3. Let $\mathbb{U}$ be a Brownian bridge process on $[0,1]$. For $0 \le t < \infty$ define a process $\mathbb{B}$ by

$$\mathbb{B}(t) \equiv (1+t)\mathbb{U}\left(\frac{t}{1+t}\right).$$

Show that $\mathbb{B}$ is a Brownian motion process on $[0,\infty)$.

Solution: Since $\mathbb{B}$ is clearly Gaussian, it suffices to show that $\mathbb{B}$ has $E\mathbb{B}(t) = 0$ and has $E(\mathbb{B}(s)\mathbb{B}(t)) = s \wedge t$ for all $0 \le s, t < \infty$. But

$$\begin{aligned}
E\mathbb{B}(t) &= (1+t)E\left\{\mathbb{U}\left(\frac{t}{1+t}\right)\right\} = (1+t)\cdot 0 = 0, \\
E\left\{\mathbb{B}(s)\mathbb{B}(t)\right\} &= (1+s)(1+t)E\left\{\mathbb{U}\left(\frac{s}{1+s}\right)\mathbb{U}\left(\frac{t}{1+t}\right)\right\} \\
&= (1+s)(1+t)\left\{\frac{s}{1+s}\wedge\frac{t}{1+t} - \frac{s}{1+s}\cdot\frac{t}{1+t}\right\} \\
&= (1+s)(1+t)\left\{\frac{s}{1+s} - \frac{s}{1+s}\cdot\frac{t}{1+t}\right\} \qquad \text{if} \quad s \le t \\
&= s(1+t) - st = s \qquad \text{if} \;\; s \le t.
\end{aligned}$$

Thus $E(\mathbb{B}(s)\mathbb{B}(t)) = s \wedge t$. It follows that $\mathbb{B}$ is Brownian motion on $[0, \infty)$.

4. PfS, Exercise 3.4, page 328: let $Z_0, Z_1, \ldots$ be i.i.d. $N(0,1)$. Let $f_j(t) \equiv \sqrt{2}\sin(j\pi t)$ for $j \ge 1$; these are orthogonal functions.
(a) Graph the first few f_j's.
(b) Verify that

$$\mathbb{U}(t) \equiv \sum_{j=1}^{\infty} Z_j f_j(t)/j\pi, \qquad 0 \le t \le 1$$

is a Brownian bridge process on $[0,1]$.
(c) Show that with $\mathbb{U}$ as defined in (b) we have

$$W^2 \equiv \int_0^1 \mathbb{U}^2(t)dt = \sum_{j=1}^{\infty} \frac{Z_j^2}{\pi^2 j^2}.$$

(This random variable gives the asymptotic null distribution of the Cramér - von Mises statistic.

Solution: (a) See the plots below:

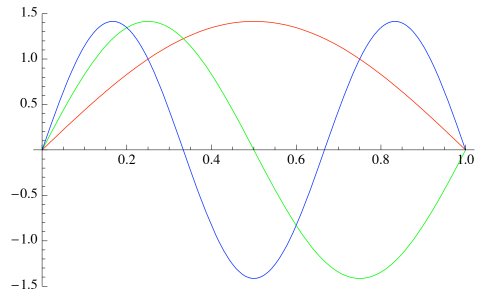

Figure 3: Sine orthogonal functions for $j = 1, 2, 3$.

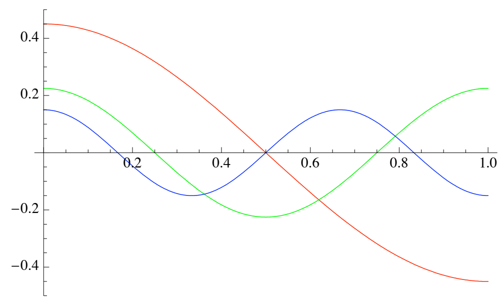

Figure 4: Cosine orthogonal functions for $j = 1, 2, 3$.

(b) Since $E(Z_j) = 0$ it follows that

$$E\mathbb{U}(t) = \sum_{j=1}^{\infty} E(Z_j)\frac{f_j(t)}{j\pi} = 0.$$

To compute the covariance, note that

$$\frac{f_j(t)}{j\pi} = \int_0^1 (1_{[0,t]}(u) - t)\sqrt{2}\cos(j\pi u)du \equiv \int_0^1 (1_{[0,t]}(u) - t)\varphi_j(u)du$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is a complete orthonormal family for the subspace of $L_2[0,1]$ consisting of all functions $x \in L_2[0,1]$ for which $\int_0^1 x(t)dt = 0$.

Then, since $E(Z_j Z_j') = 1\{j = j'\}$,

$$
\begin{aligned}
Cov[\mathbb{U}(s), \mathbb{U}(t)] &= E\{\mathbb{U}(s)\mathbb{U}(t)\} = \sum_{j=1}^{\infty}\sum_{j'=1}^{\infty} E(Z_j Z_{j'}) \frac{f_j(s) f_{j'}(t)}{j j' \pi^2} \\
&= \sum_{j=1}^{\infty} \frac{f_j(s) f_j(t)}{j^2 \pi^2} \\
&= \sum_{j=1}^{\infty} \int_0^1 (1_{[0,s]}(u) - s)\varphi_j(u)du \int_0^1 (1_{[0,s]}(u) - s)\varphi_j(u)du \\
&= \int_0^1 (1_{[0,s]}(u) - s)(1_{[0,t]}(u) - t)du = s \wedge t - st.
\end{aligned}
$$

where we used Parseval's identity in the last line.
(c) Note that

$$\mathbb{U}(t) = \sum_{j=1}^{\infty} \frac{Z_j}{j\pi} f_j(t)$$

where $\{f_j\}_{j=1}^{\infty}$ is a complete orthonormal family for the subspace of $L_2[0,1]$ consisting of all L_2 functions vanishing at 0 and 1. Thus by Parseval's identity again

$$\int_0^1 \mathbb{U}(t)^2 dt = \sum_{j=1}^{\infty} \frac{Z_j^2}{j^2 \pi^2}.$$

5. PfS, Exercise 3.1, page 327. Suppose that $Z \sim N(0,1)$ and the Brownian bridge processes $\mathbb{V}$, $\mathbb{U}^{(1)}$, and $\mathbb{U}^{(2)}$ are independent. Let $a > 0$. Show that:
(i) $\mathbb{S}(t) \equiv \mathbb{V}(t) + tZ$, $0 \le t \le 1$, is a Brownian motion.
(ii) $\mathbb{B}(t) \equiv \mathbb{S}(at)/\sqrt{a}$, $0 \le t < \infty$, is a Brownian motion.
(iii) $\mathbb{B}(t) \equiv \mathbb{S}(a+t) - \mathbb{S}(a)$, $t \ge 0$ is a Brownian motion.
(iv) $\mathbb{V}^{(1)} \equiv \sqrt{a}\mathbb{U}^{(1)} + \sqrt{1-a}\mathbb{U}^{(2)}$ is a Brownian bridge if $0 \le a \le 1$.
(v) $\mathbb{Z}(t) \equiv \left[\mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\right]/\sqrt{2}$, $0 \le t \le 1/2$ is a Brownian bridge.
(vi) $\mathbb{U}(t) = (1-t)\mathbb{S}(t/(1-t))$, $0 \le t \le 1$ is a Brownian bridge; use the LIL at infinity to show that this $\mathbb{U}$ converges to 0 at $t = 1$.

(vii) $\mathbb{B}(t) \equiv t\mathbb{S}(1/t)$, $0 \le t < \infty$, is a Brownian motion; apply the LIL of (10) to verify that these sample paths converge to 0 at $t = 0$.

Solution: (i) Note that $\mathbb{S}$ is Gaussian with $E\mathbb{S}(t) = E\mathbb{V}(t) + tE(Z) = 0 + t \cdot 0 = 0$ and

$$E\{\mathbb{S}(s)\mathbb{S}(t)\} = E\{(\mathbb{V}(s) + sZ)(\mathbb{V}(t) + tZ)\} = E\{\mathbb{V}(s)\mathbb{V}(t)\} + st = s \wedge t$$

for $0 \le s, t \le 1$.
(ii) $\mathbb{B}$ is Gaussian with $E\mathbb{B}(t) = E\mathbb{S}(at)/\sqrt{a} = 0$ and

$$E\{\mathbb{B}(s)\mathbb{B}(t)\} = E\{\mathbb{S}(as)\mathbb{S}(at)\}/a = (as \wedge at)/a = s \wedge t$$

for $0 \le s, t \le 1$.
(iii) $\mathbb{B}$ is Gaussian with $E\mathbb{B}(t) = E\{\mathbb{S}(a+t) - \mathbb{S}(a)\} = 0 - 0 = 0$ and

$$\begin{aligned} &E\{\mathbb{B}(s)\mathbb{B}(t)\} \\ &\quad = E\{(\mathbb{S}(a+s) - \mathbb{S}(a))(\mathbb{S}(a+t) - \mathbb{S}(a))\} \\ &\quad = E\{\mathbb{S}(a+s)\mathbb{S}(a+t)\} - E\{\mathbb{S}(a+s)\mathbb{S}(a)\} - E\{\mathbb{S}(a+t)\mathbb{S}(a)\} + E\{\mathbb{S}(a)^2\} \\ &\quad = (a+s) \wedge (a+t) - a - a + a \\ &\quad = a + s \wedge t - a = s \wedge t \end{aligned}$$

for $0 \le s, t \le 1$.
(iv) $\mathbb{V}^{(1)}$ is Gaussian with $E\mathbb{V}^{(1)}(t) = 0$ and

$$\begin{aligned} E\{\mathbb{V}^{(1)}(s)\mathbb{V}^{(1)}(t)\} &= (1-a)E\{\mathbb{U}^{(1)}(s)\mathbb{U}^{(1)}(t)\} + aE\{\mathbb{U}^{(2)}(s)\mathbb{U}^{(2)}(t)\} \\ &= (1-a)(s \wedge t - st) + a(s \wedge t - st) = s \wedge t - st \end{aligned}$$

for $0 \le s, t < \infty$.
(v) $\mathbb{Z}$ is Gaussian with $E\mathbb{Z}(t) = 0 + 0 = 0$ and

$$\begin{aligned} E\{\mathbb{Z}(s)\mathbb{Z}(t)\} &= E\{\left[\mathbb{U}^{(1)}(s) + \mathbb{U}^{(2)}(1-s)\right]\left[\mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\right]\}/2 \\ &= 2^{-1}\{s \wedge t - st + (1-s) \wedge (1-t) - (1-s)(1-t)\} \\ &= 2^{-1}\{s(1-t) + (1-t)(1-(1-s))\} \qquad \text{if } \ s \le t, \\ &= 2^{-1}\{s(1-t) + s(1-t)\} \qquad \text{if } \ s \le t, \\ &= s \wedge t - st \end{aligned}$$

for $0 \le s, t \le 1$.
(vi) $\mathbb{U}$ is Gaussian with $E\mathbb{U}(t) = 0$ and

$$\begin{aligned}
E\{\mathbb{U}(s)\mathbb{U}(t)\} &= (1-s)(1-t)E\left\{\mathbb{S}\left(\frac{s}{1-s}\right)\mathbb{S}\left(\frac{t}{1-t}\right)\right\} \\
&= (1-s)(1-t)\frac{s}{1-s} \wedge \frac{t}{1-t} \\
&= (1-s)(1-t)\frac{s}{1-s} \qquad \text{if } \ s \le t \\
&= (1-t)s \qquad \text{if } \ s \le t \\
&= s \wedge t - st
\end{aligned}$$

for $0 \le s, t \le 1$. As $t \to 1$, $u \equiv t/(1-t) \to \infty$ and the LIL for $\mathbb{S}$ at infinity yields

$$\begin{aligned}
&\limsup_{t\to 1} |\mathbb{U}(t)| \\
&= \limsup_{t\to 1} \, (1-t)\frac{|\mathbb{S}(t/(1-t))|}{\sqrt{2(t/(1-t))\log\log(t/(1-t))}} \cdot \sqrt{2(t/(1-t))\log\log(t/(1-t))} \\
&= \limsup_{t\to 1} \frac{|\mathbb{S}(t/(1-t))|}{\sqrt{2(t/(1-t))\log\log(t/(1-t))}} \cdot \lim_{t\to 1}(1-t)\sqrt{2(t/(1-t))\log\log(t/(1-t))} \\
&= 1 \cdot 0 = 0 \qquad \text{a.s.}
\end{aligned}$$

(vii) $\mathbb{B}$ is Gaussian with $E\mathbb{B}(t) = tE\mathbb{S}(1/t) = 0$ and

$$\begin{aligned}
E\{\mathbb{B}(s)\mathbb{B}(t)\} &= stE\{\mathbb{S}(1/s)\mathbb{S}(1/t)\} \\
&= st\{(1/s) \wedge (1/t)\} \\
&= st \cdot (1/t) \qquad \text{if } \ s \le t \\
&= s \qquad \text{if } \ s \le t = s \wedge t
\end{aligned}$$

for $0 \le s, t < \infty$. To see that $\mathbb{B}(t) \to_{a.s.} 0$ as $t \to 0$, note that by the LIL for Brownian motion $\mathbb{S}$ at infinity

$$\begin{aligned}
\limsup_{t\to 0} |\mathbb{B}(t)| &= \limsup_{t\to 0} t\frac{\mathbb{S}(1/t)}{\sqrt{2(1/t)\log\log(1/t)}} \cdot \sqrt{2(1/t)\log\log(1/t)} \\
&= \limsup_{t\to 0} \frac{\mathbb{S}(1/t)}{\sqrt{2(1/t)\log\log(1/t)}} \cdot \lim_{t\to 0} \, t\sqrt{2(1/t)\log\log(1/t)} \\
&= 1 \cdot 0 \qquad \text{a.s.}
\end{aligned}$$