Statistics 522, Problem Set 5 Solutions

Wellner; 2/21/2008

1. Exercise 11.8.7, page 55, Wellner, Chapter 11 notes. (new numbering; exercise 11.6.7, page 33, old numbering). Suppose that X and Y are independent random vectors, and that W is another random vector independent of X with $E(Y) = E(W)$ and $Cov(Y) = Cov(W)$ and satisfying $E|Y|^3 < \infty$ and $E|W|^3 < \infty$. Show that if $f \in C^3(\mathbb{R}^k)$ (define carefully what you mean by this class of functions), then

$$|Ef(X+Y) - Ef(X+W)| \le C\{E|Y|^3 + E|W|^3\}$$

where C is a constant depending only on third derivatives of f.

Solution: First, note that we can write

$$f(x+y) = f(x) + y'\dot{f}(x) + \frac{1}{2}y'\ddot{f}(x)y + \frac{1}{6}\sum_{i,j,l=1}^{k} y_i y_j y_l \dddot{f}_{i,j,l}(x^*)$$

where $|x^* - x| \le |x + y - x| = |y|$. Thus, noting that $EY = EW$ and $Cov(Y) = Cov(W)$ together imply that $E(YY') = E(WW')$, it follows by using the independence of X and Y and independence of X and W that

$$\begin{aligned}
&|Ef(X+Y) - Ef(X+W)| \\
&= \Big|E(Y'\dot{f}(X)) - E(W'\dot{f}(X)) + \frac{1}{2}E(Y'\ddot{f}(X)Y) - \frac{1}{2}E(W'\ddot{f}(X)W) \\
&\quad + \frac{1}{6}E\{\sum_{i,j,l=1}^{k} Y_iY_jY_l\dddot{f}_{i,j,l}(X^*)\} - \frac{1}{6}E\{\sum_{i,j,l=1}^{k} W_iW_jW_l\dddot{f}_{i,j,l}(X^{**})\}\Big| \\
&\le |(EY - EW)'E\dot{f}(X) + 2^{-1}\{E(Y'\ddot{f}(X)Y) - E(W'\ddot{f}(X)W)\}| \\
&\quad + \frac{1}{6}\left\{|E\{\sum_{i,j,l=1}^{k} Y_iY_jY_l\dddot{f}_{i,j,l}(X^*)\}| + |E\{\sum_{i,j,l=1}^{k} W_iW_jW_l\dddot{f}_{i,j,l}(X^{**})\}|\right\} \\
&\le |0+0| + \frac{1}{6}\|\dddot{f}\|_\infty\left\{E\left(\sum_{i=1}^{k}|Y_i|\right)^3 + E\left(\sum_{i=1}^{k}|W_i|\right)^3\right\} \\
&\le \frac{k^{3/2}\|\dddot{f}\|_\infty}{6}\left\{E\|Y\|_2^3 + E\|W\|_2^3\right\}
\end{aligned}$$

where

$$\|\dddot{f}\|_\infty \equiv \max_{1\le i,j,l\le k} \sup_{x\in\mathbb{R}^k} |\dddot{f}_{i,j,l}(x)|.$$

In the next to last inequality we used $tr(v'Av) = tr(Avv')$ and hence

$$\begin{aligned} E\{Y'\ddot{f}(X)Y\} &= E\{tr(Y'\ddot{f}(X)Y)\} = E\{tr(\ddot{f}(X)YY')\} \\ &= tr(E\{\ddot{f}(X)YY'\}) = tr(E\{\ddot{f}(X)\}E(YY')) \\ &= tr(E\{\ddot{f}(X)\}E(WW')) = tr(E\{\ddot{f}(X)WW'\}) \\ &= E\{tr\ddot{f}(X)WW')\} = E\{tr(W'\ddot{f}(X)W)\} \\ &= E\{W'\ddot{f}(X)W\}, \end{aligned}$$

and in the last inequality we used

$$\sum_{i=1}^k |Y_i| \le \sqrt{k}\|Y\|_2 \qquad \text{by the Cauchy-Schwarz inequality.}$$

Thus the claimed inequality holds with $C \equiv k^{3/2}\|\dddot{f}\|_\infty/6$.

2. Exercise 11.8.8, page 55, Wellner, Chapter 11 notes. Let Y be a random vector in $\mathbb{R}^k$ with $\mu = E(Y)$ and $\Sigma = Cov(Y) = E\{(Y-\mu)(Y-\mu)'\}$. Thus we can write $\Sigma = A\Lambda^2 A'$ where A is an orthogonal matrix (so $AA' = I$) and Λ is diagonal with each diagonal entry non-negative. Define $B = A\Lambda$. Let Z be a random vector with independent $N(0,1)$ coordinates; thus $Z \sim N_k(0, I)$.
(a) Show that $|\mu| \le E\|Y\|$. [Hint: note that $u'Y \le |Y|$ for all unit vectors u, and in particular for $u = \mu/|\mu|$.
(b) Show that $E|BZ|^3 = E|\Lambda Z|^3 \le (\text{trace}(\Sigma))^{3/2} E|Z_1|^3$.
(c) Show that $E|\mu + BZ|^3 \le 8E|Y|^3 + 8(E|Y|^2)^{3/2}E|Z_1|^3$. Can the factor 8 be improved to 4?

Solution: (a) By the hint $u'E(Y) \le E|Y|$ for all unit vectors u, and hence with $u = \mu/|\mu|$ it follows that

$$E|Y| \ge \mu' E(Y)/|\mu| = \mu'\mu/|\mu| = |\mu|^2/|\mu| = |\mu|.$$

(b) Now $BZ = A\Lambda Z$, and hence

$$|BZ|^2 = \|BZ\|_2^2 = Z'\Lambda A' A\Lambda Z = Z'\Lambda^2 Z = \sum_{i=1}^k \lambda_i^2 Z_i^2 = \|\Lambda Z\|_2^2.$$

Thus it follows that

$$\begin{aligned}
|BZ|^3 &= \left(\sum_{i=1}^k \lambda_i^2 Z_i^2\right)^{3/2} = \left(\frac{\sum_{i=1}^k \lambda_i^2 Z_i^2}{\sum_{i=1}^k \lambda_j^2}\right)^{3/2} \cdot \left(\sum_{j=1}^k \lambda_j^2\right)^{3/2} \\
&\equiv \left(\sum_{i=1}^k p_i Z_i^2\right)^{3/2} \cdot \left(\sum_{j=1}^k \lambda_j^2\right)^{3/2} \\
&\qquad \text{where } \ p_i \equiv \lambda_i^2 / \sum_1^k \lambda_j^2 \ \text{ satisfy } \ \sum_{i=1}^k p_i = 1 \\
&\le \sum_{i=1}^k p_i |Z_i|^3 \cdot \left(\sum_{j=1}^k \lambda_j^2\right)^{3/2} \quad \text{by Jensen's inequality.}
\end{aligned}$$

Therefore

$$E|BZ|^3 = E\|\Lambda Z\|_2^3 \le \left(\sum_{j=1}^k \lambda_j^2\right)^{3/2} E|Z_1|^3 = (\mathrm{tr}(\Sigma))^{3/2} E|Z_1|^3$$

where the last equality uses

$$\mathrm{tr}(\Sigma) = \mathrm{tr}(BB') = \mathrm{tr}(A\Lambda^2 A') = \mathrm{tr}(\Lambda^2 A' A) = \mathrm{tr}(\Lambda^2).$$

(c) First, $|\mu + BZ| \le |\mu| + |BZ|$. Then by the C_r-inequality with $r = 3$,

$$\begin{aligned}
E|\mu + BZ|^3 &\le E\{(|\mu| + |BZ|)^3\} \le 2^{3-1}\left\{|\mu|^3 + E|BZ|^3\right\} \\
&\le 4\left\{(E|Y|)^3 + [\mathrm{tr}(\Sigma)]^{3/2} E|Z_1|^3\right\} \qquad \text{by (a) and (b)} \\
&\le 4\{E|Y|^3 + (E|Y|^2)^{3/2} E|Z_1|^{3/2}
\end{aligned}$$

by Jensen's inequality and since $\mathrm{tr}(\Sigma) = \sum_{j=1}^k Var(Y_j) \le \sum_{j=1}^k E(Y_j^2) = E|Y|^2$.

3. Exercise 11.8.9, page 55, Wellner, Chapter 11, notes. Use the Cramér - Wold device to prove the classical multivariate CLT from the univariate CLT.

Solution: Let $Z_n \equiv \sqrt{n}(\overline{X}_n - \mu)$ where $\overline{X}_n = n^{-1}(X_1 + \cdots + X_n)$ and $X_1, \ldots, X_n$ are i.i.d. in $\mathbb{R}^k$ with $E(X_1) = \mu$, $E(X_1^T X_1) = E\{|X_1|^2\} <$

∞, and $Cov(X) = E(X-\mu)(X-\mu)^T = \Sigma$. We want to show that $Z_n \to_d N_k(0,\Sigma)$ in $\mathbb{R}^k$. Let $a \in \mathbb{R}^k$ and set $V_n \equiv a^T Z_n$. Note that

$$V_n = n^{-1/2} \sum_{i=1}^n a^T (X_i - \mu) \equiv n^{-1/2} \sum_{i=1}^n i$$

where $Y_1, \ldots, Y_n$ are i.i.d. real-valued random variables with $EY_1 = 0$, $Var(Y_1) = a^T \Sigma a$. Thus by the univariate classical CLT $V_n \to_d N_1(0, a^T\Sigma a)$. But since $a^T Z \sim N_1(0, a^T \Sigma a)$ if $Z \sim N_k(0,\Sigma)$, it follows that

$$V_n = a^T Z_n \to_d a^T Z.$$

Thus the Cramér - Wold theorem implies that $Z_n \to_d Z \sim N_k(0,\Sigma)$.

4. PfS, exercise 11.8.4, page 317. Suppose that the random variable $\log X \sim N(0,1)$; thus

$$f_X(x) = x^{-1} \frac{1}{\sqrt{2\pi}} e^{-(\log x)^2/2} 1_{(0,\infty)}(x).$$

For each $-1 \le a \le 1$ let Y_a have the density function

$$f_a(y) = f_X(y)\,[1 + a\sin(2\pi \log y)]\,1_{(0,\infty)}(y).$$

Show that X and each Y_a have exactly the same moments.

Solution: First, the moments of X are given as follows:

$$\begin{aligned} EX^k &= \int_0^\infty x^k f_X(x)dx = \int_0^\infty x^{k-1} e^{-(\log x)^2/2} (2\pi)^{-1/2} dx \\ &= \int_{-\infty}^\infty e^{kv} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv \\ &= \exp(k^2/2). \end{aligned}$$

Now we compute the moments of Y_a for fixed $a \in [-1,1]$: writing

$\phi(v) \equiv (2\pi)^{-1/2} e^{-v^2/2}$,

$$
\begin{aligned}
EY_a^k &= \int_0^\infty y^k f_X(y) \left[1 + a \sin(2\pi \log y)\right] dy \\
&= \int_{-\infty}^\infty e^{kv} \phi(y) \left[1 + a \sin(2\pi v)\right] dv \\
&= \exp(k^2/2) + a \int_{-\infty}^\infty e^{kv} \sin(2\pi v) \phi(v) dv \\
&= \exp(k^2/2) \left\{ 1 + a \int_{-\infty}^\infty \sin(2\pi v) \frac{1}{\sqrt{2\pi}} \exp(-(v-k)^2/2) dv \right\} \\
&= \exp(k^2/2) + 0 = \exp(k^2/2).
\end{aligned}
$$

Here the zero in the last line holds since

$$
\begin{aligned}
&\int_{-\infty}^\infty \sin(2\pi v) \frac{1}{\sqrt{2\pi}} \exp(-(v-k)^2/2) dv \\
&= \int_{-\infty}^\infty \sin(2\pi(z+k)) \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) dz \\
&= \int_{-\infty}^\infty \sin(2\pi z) \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) dz \\
&= 0
\end{aligned}
$$

where the next to last equality holds by the trigonometric formula $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and hence

$$\sin(2\pi(z+k)) = \sin(2\pi z)\cos(2\pi k) + \cos(2\pi z)\sin(2\pi k) = \sin(2\pi z),$$

and the last equality holds since $\sin(2\pi z)$ is an odd function of z while the standard normal density is an even function of z. Thus $EY_a^k = EX^K$ for all $k \geq 1$ and all $a \in [-1, 1]$. Thus the moment sequence does not characterize the distribution in general.

5. **Optional bonus problem:**
(a) PfS, exercise 11.8.5, page 317: Show that if $\limsup |\mu_k|^{1/k}/k < \infty$, then at most one distribution function F can possess the moment values $\mu_k = \int x^k dF(x)$.
(b) PfS, exercise 11.8.6, page 318: Show that the standard normal distribution $N(0,1)$ is uniquely determined by its moments.