Statistics 522, Problem Set 4 Solutions

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1. Suppose that $Y_1, \ldots, Y_n$ be independent Bernoulli (1/2) random variables so that $X_j = 2Y_j - 1$ are independent Rademacher random variables. with $EX_j = 0$ and $Var(X_j) = 1$. Let $S_n = X_1 + \cdots + X_n$, and $\widetilde{S}_n = S_n/\sqrt{n}$, and define $F_n(x) = P(\widetilde{S}_n \le x)$. Show that

$$\limsup_{n\to\infty} \left\{ \sqrt{n} \|F_n - \Phi\|_\infty \right\} \ge \frac{1}{\sqrt{2\pi}}.$$

Hint: Take $n = 2m+1$; take $t_m = -(2m+1)^{-1/2}$; note that $\sum_1^n Y_j \sim$Binomial$(n, 1/2)$; show that $P(Bin(2m+1, 1/2) \le m) = 1/2$; and hence that $F_{2m+1}(t_m) - \Phi(t_m) = \int_{t_m}^0 \phi(y)dy$.

Solution: Note that

$$\begin{aligned} F_n(x) &= P(\widetilde{S}_n \le x) = P(\sum_{i=1}^n X_i \le x\sqrt{n}) \\ &= P(\sum_{i=1}^n (2Y_i - 1) \le x\sqrt{n}) \\ &= P(2\text{Binomial}(n, 1/2) - n \le x\sqrt{n}) \end{aligned}$$

For $n = 2m+1$ (i.e. odd) and $x = t_m \equiv -(2m+1)^{-1/2}$ it follows that

$$\begin{aligned} F_{2m+1}(t_m) &= P(2\text{Binomial}(2m+1, 1/2) - 2m + 1 \le -1) \\ &= P(\text{Binomial}(2m+1, 1/2) \le m) \\ &= 1/2 = P(\text{Binomial}(2m+1, 1/2) \ge m+1) \end{aligned}$$

by symmetry. Therefore

$$\begin{aligned} F_{2m+1}(t_m) - \Phi(t_m) &= 1/2 - \int_{-\infty}^{t_m} \phi(y)dy = \int_{-\infty}^0 \phi(y)dy - \int_{-\infty}^{t_m} \phi(y)dy \\ &= \int_{t_m}^0 \phi(y)dy, \end{aligned}$$

and hence

$$\begin{aligned}
\sqrt{2m+1}\|F_{2m+1}-\Phi\|_\infty &\geq \sqrt{2m+1}|F_{2m+1}(t_m)-\Phi(t_m)| \\
&= \sqrt{2m+1}\int_{t_m}^{0}\phi(y)dy \\
&= \frac{1}{\epsilon}\int_0^{\epsilon}\phi(y)dy \quad \text{with} \quad \epsilon = -t_m \to 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\limsup_{n\to\infty}\sqrt{n}\|F_n-\Phi\|_\infty &\geq \limsup_{m\to\infty}\sqrt{2m+1}\|F_{2m+1}-\Phi\|_\infty \\
&= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^{\epsilon}\phi(y)dy \\
&= \lim_{\epsilon\downarrow 0}\phi(\epsilon) = (2\pi)^{-1/2}.
\end{aligned}$$

2. Stroock's chapter II, page 64: verify the claim that the bound in Lindeberg's CLT can be taken to be the expression in (2.1.13) of Stroock when the X_{ni}'s all have finite third moments.

   Solution: As in the proof of Theorem 11.6.1 (Lindeberg's theorem),

$$\begin{aligned}
\Delta &\equiv |E\varphi(\widetilde{S}_n) - E\varphi(Z)| \\
&\leq \sum_{i=1}^{n}\left\{E|R(U_{ni};\widetilde{X}_{ni})| + E|R(U_{ni};\widetilde{Y}_{ni})|\right\}
\end{aligned}$$

   where

$$R(x;y) \leq \|\varphi'''\|_\infty\frac{1}{6}|y|^3.$$

   Thus

$$E|R(U_{ni};\widetilde{X}_{ni})| \leq \frac{1}{6}\|\varphi'''\|_\infty E|\widetilde{X}_{ni}|^3 = \frac{1}{6}\|\varphi'''\|_\infty\frac{\tau_{ni}^3}{\sigma_n^3},$$

   and, as before,

$$E|R(U_{ni};\widetilde{Y}_{ni})| \leq \frac{1}{6}\|\varphi'''\|_\infty E|Z|^3\frac{\sigma_{ni}^3}{\sigma_n^3} \leq \frac{1}{6}\|\varphi'''\|_\infty E|Z|^3\frac{\tau_{ni}^3}{\sigma_n^3},$$

where the second inequality follows from Liapunov's inequality and where

$$E|Z|^3 = 2\int_0^\infty z^3\phi(z)dz = 2\sqrt{2/\pi}.$$

Therefore, summing on i and using the notation

$$\rho_n \equiv \sum_{i=1}^n E|X_{ni}|^3/\sigma_n^3,$$

$$\Delta \le \|\varphi'''\|_\infty \rho_n \frac{1}{6}\left(1+2\sqrt{2/\pi}\right)$$

where

$$\frac{1}{6}\left(1+2\sqrt{2/\pi}\right) \doteq .432628\ldots < 2/3.$$

Thus Stroock's claimed bound holds; in fact Stroock's 2/3 can be reduced to the constant above which is less than .45.

3. (Stein's method for Poisson random variables). Suppose that $X \sim$ Poisson(λ), and write P_λ for the corresponding Poisson probability measure on $\mathbb{Z}^+$ and E_λ for expectation under P_λ. Show that the following identity holds: for any bounded function $g : \mathbb{Z}^+ \mapsto \mathbb{R}$,

$$E_\lambda\{\lambda g(X+1) - Xg(X)\} = 0.$$

Solution: Suppose $X \sim$ Poisson(λ) and g is bounded. Then we compute

$$\begin{aligned}
&E\{\lambda g(X+1) - Xg(X)\} \\
&= \sum_{x=0}^\infty \lambda g(x+1)e^{-\lambda}\frac{\lambda^x}{x!} - \sum_{x=0}^\infty xg(x)e^{-\lambda}\frac{\lambda^x}{x!} \\
&= \sum_{x=0}^\infty (x+1)g(x+1)e^{-\lambda}\frac{\lambda^{x+1}}{(x+1)!} - \sum_{x=0}^\infty xg(x)e^{-\lambda}\frac{\lambda^x}{x!} \\
&= \sum_{y=1}^\infty yg(y)e^{-\lambda}\frac{\lambda^y}{y!} - \sum_{x=1}^\infty xg(x)e^{-\lambda}\frac{\lambda^x}{x!} = 0.
\end{aligned}$$

4. (Stein's method for Poisson random variables, continued). Let $A \subset \mathbb{Z}^+$. Consider the bounded function $\varphi(z) = 1_A(z)$ for $z \in \mathbb{Z}^+$ and its Poisson-centered version $\widetilde{\varphi}(z) = \varphi(z) - E_\lambda \varphi(X)$. Suppose that $g = g_{\lambda,A}$ is constructed to solve the equation

$$\lambda g(z+1) - z g(z) = \widetilde{\varphi}(z) = 1_A(z) - P_\lambda(A). \tag{1}$$

Show that the solution of the equation in the last display is given by

$$\begin{aligned} g(z+1) &= \lambda^{-z-1} z! e^\lambda \{P_\lambda(A \cap U_z) - P_\lambda(A) P_\lambda(U_z)\} \\ &= \lambda^{-z-1} z! e^\lambda \{P_\lambda(A \cap U_z) P_\lambda(U_z^c) - P_\lambda(A \cap U_z^c) P_\lambda(U_z)\} \end{aligned}$$

for $z \geq 0$, $z \in \mathbb{Z}^+$ where $U_z \equiv \{0, \ldots, z\}$.

Solution: With $g(z+1)$ as given in the first line of the last display, we find that

$$\begin{aligned} &\lambda g(z+1) - z g(z) \\ &= \lambda^{-z} z! e^\lambda \{P_\lambda(A \cap U_z) - P_\lambda(A) P_\lambda(U_z)\} \\ &\qquad - \lambda^{-z} z! e^\lambda \{P_\lambda(A \cap U_{z-1}) - P_\lambda(A) P_\lambda(U_{z-1})\} \\ &= \frac{1}{P_\lambda(\{z\})} \{P_\lambda(A \cap U_z) - P_\lambda(A \cap U_{z-1}) - P_\lambda(A)(P_\lambda(U_z) - P_\lambda(U_{z-1}))\} \\ &= \frac{1}{P_\lambda(\{z\})} \{P_\lambda(A \cap \{z\}) - P_\lambda(A) P_\lambda(\{z\})\} \\ &= 1_A(z) - P_\lambda(A) \end{aligned}$$

where the next to last equality holds because $U_z = U_{z-1} + \{z\}$ (disjoint union) and hence

$$\begin{aligned} &P_\lambda(U_z) = P_\lambda(U_{z-1}) + P_\lambda(\{z\}), \quad \text{and} \\ &P_\lambda(A \cap U_z) = P_\lambda(A \cap U_{z-1}) + P_\lambda(A \cap \{z\}). \end{aligned}$$

The second identity follows by writing $P_\lambda(A) = P_\lambda(A \cap U_z) + P_\lambda(A \cap U_z^c)$ so that

$$\begin{aligned} &\{P_\lambda(A \cap U_z) - P_\lambda(A) P_\lambda(U_z)\} \\ &= P_\lambda(A \cap U_z) - \{P_\lambda(A \cap U_z) + P_\lambda(A \cap U_z^c)\} P_\lambda(U_z) \\ &= P_\lambda(A \cap U_z) P_\lambda(U_z^c) - P_\lambda(A \cap U_z^c) P_\lambda(U_z). \end{aligned}$$

5. (Stein's method for Poisson random variables, continued). Suppose that $W_n = \sum_{i=1}^n X_i$ where $X_1, \ldots, X_n$ are independent, $X_i \sim$ Bernoulli(p_i). Let $\lambda \equiv \lambda_n = E(W_n) = \sum_{i=1}^n p_i$. Use the identity of the previous problem to show that for any set $A \subset \mathbb{Z}^+$

$$|P(W \in A) - P_\lambda(A)| \le \left\{ \begin{array}{l} 2 \sup_z |g_{\lambda,A}(z)| \sum_{i=1}^n p_i^2 \\ \sup_z |g_{\lambda,A}(z+1) - g_{\lambda,A}(z)| \sum_{i=1}^n p_i^2 . \end{array} \right.$$

Hint: Use the identity in the previous problem, then note that $E\{X_i g(W)\} = E\{X_i g(W_i + 1)\} = p_i E g(W_i + 1)$ where $W_i = \sum_{j \ne i} X_j$. Furthermore, note that W and W_i are equal unless $X_i = 1$, which occurs with probability p_i.

Solution: This is as in the notes: since the identity $\lambda g(z+1) - z g(z) = 1_A(z) - P_\lambda(A)$ holds for any set $A \subset \mathbb{Z}^+$, it follows that

$$\begin{aligned} & P(W \in A) - P_\lambda(A) \\ &= E\{1_A(E) - P_\lambda(Z)\} \\ &= E\{\lambda g(W+1) - W g(W)\} \\ &= E\{\sum_{i=1}^n [p_i g(W+1) - X_i g(W)]\} \\ &= \sum_{i=1}^n p_i E\{g(W+1) - g(W_i + 1)\} \qquad (2) \\ &= \sum_{i=1}^n p_i E\big[\{g(W+1) - g(W_i+1)\} 1_{[X_i = 1]}\big] \qquad (3) \end{aligned}$$

where (3) holds since $E\{X_i g(W)\} = E\{X_i g(W_i + 1)\} = p_i E g(W_i + 1)$ where $W_i \equiv \sum_{j \ne i, j=1}^n$, and (3) holds because $W = W_i$ unless $X_i = 1$, an event with probability p_i. The claimed bounds follow immediately from (3).

6. (Stein's method for Poisson random variables, continued). It is shown in Barbour, Holst, and Janson (1992) that for $g = g_{\lambda,A}$ solving (1), the following bounds hold:

$$\|g\|_\infty \le \min\{1, \lambda^{-1/2}\}, \qquad \|\Delta g\|_\infty \le \lambda^{-1}(1 - e^{-\lambda}).$$

Use these bounds to show that

$$d_{TV}(P(W_n \in \cdot), P_\lambda) \leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{i=1}^{n} p_i^2,$$

which improves substantially on the bound $\sum_{i=1}^{n} p_i^2$ established in problem 6, Math/Stat 521 homework set 6, via coupling.

Solution: This follows almost immediately from the preceding problem and the given bound for $\|\Delta g\|$:

$$\begin{aligned} d_{TV}(P_W, P_\lambda) &= \sup_{A \subset \mathbb{Z}^+} |P(W_n \in A) - P_\lambda(A)| \\ &\leq \sup_{A \subset \mathbb{Z}^+} \sup_{z} |g_{\lambda,A}(z+1) - g_{\lambda,A}(z)| \sum_{i=1}^{n} p_i^2 \\ &\leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{i=1}^{n} p_i^2. \end{aligned}$$