**Statistics 522, Problem Set 3 Solutions**

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1. Exercise 11.6.1, page 34, Wellner, Chapter 11 notes. Prove the equivalence of (i) and (ii) in Proposition 11.2.2.

   **Solution:** Suppose that (ii) holds. Let

$$h(x) = \begin{cases} 1, & x < 0, \\ 1 - x, & 0 \le x \le 1, \\ 0, & x > 1, \end{cases}$$

   and for fixed $x \in \mathbb{R}$ and $\epsilon > 0$, consider the functions

$$f_{\epsilon,x}(y) = h\left(\frac{y-x}{\epsilon}\right),$$
$$g_{\epsilon,x}(y) = h\left(\frac{y-x+\epsilon}{\epsilon}\right).$$

   Note that $f_{\epsilon,x}$ and $g_{\epsilon,x}$ are continuous and bounded (since their range is $[0,1]$. Moreover

$$1_{(-\infty,x-\epsilon]}(y) \le g_{\epsilon,x}(y) \le 1_{(-\infty,x]}(y) \le f_{\epsilon,x}(y) \le 1_{(-\infty,x+\epsilon]}(y) \qquad \text{for all } y \in \mathbb{R}.$$

   Thus by (ii) $Ef_{\epsilon,x}(X_n) \to Ef_{\epsilon,x}(X)$ and $Eg_{\epsilon,x}(X_n) \to Eg_{\epsilon,x}(X)$. Therefore

$$F_n(x) = E1_{(-\infty,x]}(X_n) \le Ef_{\epsilon,x}(X_n) \to Ef_{\epsilon,x}(X) \le E1_{(-\infty,x+\epsilon]}(X) = F(x+\epsilon),$$

   and, similarly,

$$F(x-\epsilon) \le Eg_{\epsilon,x}(X) \leftarrow Eg_{\epsilon,x}(X_n) \le E1_{(-\infty,x]}(X_n) = F_n(x).$$

   Therefore we conclude that

$$F(x-\epsilon) \le \liminf F_n(x) \le \limsup F_n(x) \le F(x+\epsilon)$$

for every $\epsilon > 0$. Letting $\epsilon \to 0$ and assuming that $x$ is a continuity point of $F$, we conclude that $F_n(x) \to F(x)$; i.e. (i) holds.

For the reverse implication (i) implies (ii) (which is also solved via the Skorokhod theorem on PfS, page 53), choose $I = [a, b]$ with $a, b \in C_F$ such that $P_F(I^c) = F(a) + (1 - F(b)) \le \epsilon$. Let $a = a_0 < a_1 \cdots < a_m = b$ define a partition of $\{I_j = (a_{j-1}, a_j] : 1 \le j \le m\}$ of $[a, b]$ with $a_j \in C_F$ for all $0 \le j \le m$. Based on this partition define approximating functions $f_m^+$ and $f_m^-$ by

$$f_m^+(x) \equiv \sum_{j=1}^m \sup_{x \in I_j}(f(x)) 1_{I_j}(x),$$
$$f_m^-(x) \equiv \sum_{j=1}^m \inf_{x \in I_j}(f(x)) 1_{I_j}(x).$$

Then

$$\int_I f_m^- dF_n \le \int_I f dF_n \le \int_I f_m^+ dF_n.$$

The left and right sides in the last display converge, in view of our hypothesis, and hence

$$\int_I f_m^- dF \le \liminf_{n\to\infty} \int_I f dF_n \le \limsup_{n\to\infty} \int_I f dF_n \le \int_I f_m^+ dF.$$

Now let $\max_{j \le m} |a_j - a_{j-1}| \to 0$. Then, by choosing the partitions to be nested,

$$f_m^-(x) \uparrow f(x), \qquad \text{and} \qquad f_m^+(x) \downarrow f(x).$$

By the dominated convergence this yields

$$\lim_m \int_I f_m^+ dF = \int_I f dF \qquad \text{and} \qquad \lim_m \int_I f_m^- dF = \int_I f dF.$$

Thus we conclude that

$$\begin{aligned} \limsup_{n\to\infty} \left| \int f dF_n - \int f dF \right| &\le \limsup_{n\to\infty} \left| \int_I f dF_n - \int_I f dF \right| + \|f\|_\infty \cdot 2\epsilon \\ &= 2\epsilon \|f\|_\infty. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this yields $\int f dF_n \to \int f dF$ for all bounded and continuous $F$.

2. Exercise 11.6.2, page 34, Wellner, Chapter 11 notes. Suppose that $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$ where both $\mu$ and $\sigma^2$ are finite. Suppose that $Z \sim P_0$ on $\mathbb{R}$.
   (a) Show that $X_n \stackrel{d}{=} \mu_n + \sigma_n Z \to \mu + \sigma Z \stackrel{d}{=} X$.
   (b) Show that for $f \in BL(\mathbb{R})$

$$|Ef(X_n) - Ef(X)| \le \|f\|_{BL} E\{1 \wedge (|\mu_n - \mu| + |\sigma_n - \sigma||Z|)\}.$$

**Solution:** (b) For $f \in BL(\mathbb{R})$ it follows that $|f(y)-f(x)| \le \|f\|_{BL}\{1\wedge |y-x|\}$ for all $x, y \in \mathbb{R}$; recall the inequality before Definition 1.4 on page 4. Thus

$$|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \le \|f\|_{BL}\{1 \wedge |\mu_n - \mu + (\sigma_n - \sigma)Z|\};$$

This implies that

$$\begin{aligned} |Ef(X_n) - Ef(X)| &= |Ef(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\ &\le E|f(\mu_n + \sigma_n Z) - f(\mu + \sigma Z)| \\ &\le \|f\|_{BL} E\{1 \wedge |\mu_n - \mu| + |\sigma_n - \sigma||Z|\}. \end{aligned}$$

(a) Since $\mu_n - \mu \to 0$ and $\sigma_n - \sigma \to 0$, it follows by the dominated convergence theorem (with dominating function 1) that the right side of the last display converges to 0. Thus $Ef(X_n) \to Ef(X)$ for all $f \in BL(\mathbb{R})$; therefore $X_n \to_d X$ by the portmanteau theorem.

3. Exercise 11.6.4, page 34, Wellner, Chapter 11 notes. Give a direct proof of the equivalence of (i) and (iv) in Proposition 2.2. Hint: Consider the functions $\psi_\epsilon(y) \equiv \psi(y/\epsilon)$ where $\psi$ is defined as follows: $\psi(y) = 1$ if $y \le 0$, $\psi(y) = 0$ if $y \ge 1$, and

$$\psi(y) = \frac{\int_y^1 \exp(-1/(u(1-u)))du}{\int_0^1 \exp(-1/(u(1-u)))du} \qquad \text{for } \ 0 \le y \le 1.$$

**Solution:** Suppose that (iv) holds. Then, in particular, $E\psi_\epsilon(X_n - x) \to E\psi_\epsilon(X - x)$ for every $\epsilon > 0$ and $x \in \mathbb{R}$. But then since

$$1_{(-\infty, x-\epsilon]}(y) \le \psi_\epsilon(y - x + \epsilon) \le 1_{(-\infty, x]}(y) \le \psi_\epsilon(y - x) \le 1_{(-\infty, x+\epsilon]}(y)$$

for all $y \in \mathbb{R}$, it follows that

$$F_n(x) = E1_{(-\infty,x]}(X_n) \leq E\psi_\epsilon(X_n - x)$$

and hence

$$\limsup_{n\to\infty} F_n(x) \leq \limsup_n E\psi_\epsilon(X_n - x) = E\psi_\epsilon(X - x) \leq E1_{(-\infty,x+\epsilon]}(X) = F(x+\epsilon).$$

and, on the other hand,

$$F(x-\epsilon) \leq E\psi_\epsilon(X - x + \epsilon) = \liminf_n E\psi_\epsilon(X_n - x + \epsilon) \leq \liminf F_n(x).$$

Putting these together yields

$$F(x-\epsilon) \leq \liminf F_n(x) \limsup F_n(x) \leq F(x+\epsilon)$$

for every $\epsilon > 0$. For $x \in C_R$, the two extremes both converge to $F(x)$ as $\epsilon \to 0$; thus $F_n \to_d F$, and (i) holds.

To see the reverse implication, suppose that (i) holds. Note that for $f \in C^\infty(\mathbb{R})$ and $a, b$ continuity points of $F$ with $F(a) < \epsilon$ and $1 - F(b-) \leq \epsilon$ we can write

$$\begin{aligned}
&|Ef(X_n) - Ef(X)| \\
&= \left| \int f d(F_n - F) = \int_{(-\infty,a]} f d(F_n - F) + \int_{(a,b]} f d(F_n - F) + \int_{(b,\infty)} f d(F_n - F) \right| \\
&\leq \|f\|_\infty \{F_n(a) + F(a)) + (1 - F_n(b-)) + (1 - F(b-))\} + \left| \int_{(a,b]} f d(F_n - F) \right| \\
&\leq \|f\|_\infty \{F_n(a) + F(a)) + (1 - F_n(b-)) + (1 - F(b-))\} \\
&\qquad + |f(b)(F_n(b) - F(b)) - f(a)(F_n(a) - F(a))| \\
&\qquad + \left| \int_{[a,b]} f'(x)(F_n - F)(x) dx \right|
\end{aligned}$$

where $f'$ is uniformly continuous on the compact set $[a, b]$ and $F_n(x) \to F(x)$ for Lebesgue a.e. $x$ (since $F$ has at most countably many points of discontinuity). Since $F_n(a) \to F(a)$ and $1 - F_n(b-) \to 1 - F(b)$ by our hypothesis, it follows by using the dominated convergence theorem on the last term (with dominating function $2\|f'\|_\infty$) that

$$\limsup_n |Ef(X_n) - Ef(X)| \quad \leq \quad 4\|f\|_\infty \epsilon + 0;$$

i.e. (iv) holds.

4. Exercise 11.6.5, page 34, Wellner, Chapter 11 notes. Show that $\lambda(F, G)$ as defined in proposition 2.3 is a metric and that the space of all distribution functions under $\lambda$ is a complete separable metric space. Also $F_n \to F$ as $n \to \infty$ if and only if $\lambda(F_n, F) \to 0$.

**Solution:**
• First, $\lambda$ is a metric:
(a) $\lambda(F, G) = \lambda(G, F)$: this follows since

$$\begin{aligned} &\inf\{\epsilon > 0 : \; F(x-\epsilon) - \epsilon \le G(x) \le F(x+\epsilon) + \epsilon \text{ for all } \; x \in \mathbb{R}\} \\ &\qquad = \inf\{\epsilon > 0 : \; G(y-\epsilon) - \epsilon \le F(y) \le G(y+\epsilon) + \epsilon \; \text{ for all } \; y \in \mathbb{R}\} \end{aligned}$$

by taking $y = x - \epsilon$ and $y = x + \epsilon$.
(b) $\lambda(F, G) = 0$ if and only if $F = G$: if $G = F$, then

$$\lambda(F, F) = \inf\{\epsilon > 0 : \; F(x-\epsilon)-\epsilon \le F(x) \le F(x+\epsilon)+\epsilon \text{ for all } \; x \in \mathbb{R}\} = 0$$

is clear. On the other hand, if $\lambda(F, G) = 0$, then

$$G(x) \le F(x+\epsilon) + \epsilon \quad \text{for all } \epsilon > 0$$

which yields $G(x) \le F(x)$ for all $x$ by right continuity of $F$. Reversing the roles of $F$ and $G$ using (a) gives $F(x) \le G(x)$ for all $x$ and hence $F(x) = G(x)$ for all $x$; i.e. $F = G$.
(c) $\lambda(F, H) \le \lambda(F, G) + \lambda(G, H)$: To see this, consider the sets

$$\begin{aligned} A &\equiv \{\epsilon > 0 : \; F(x-\epsilon) - \epsilon \le G(x) \le F(x+\epsilon) + \epsilon \; \text{ for all } \; x \in \mathbb{R}\}, \quad \text{and} \\ B &\equiv \{\delta > 0 : \; G(x-\delta) - \delta \le H(x) \le G(x+\delta) + \delta \; \text{ for all } \; x \in \mathbb{R}\}. \end{aligned}$$

Then for $\epsilon \in A$ and $\delta \in B$,

$$\begin{aligned} F(x-\epsilon-\delta) - \epsilon - \delta \;&\le\; G(x-\delta) - \delta \\ &\le\; H(x) \le G(x+\delta) + \delta \le F(x+\epsilon+\delta) + \epsilon + \delta \end{aligned}$$

for all $x \in \mathbb{R}$. This yields the claim.

• Now we show that $F_n \to F$ if and only if $\lambda(F_n, F) \to 0$.
(a) Suppose that $F_n \to F$. Fix $\epsilon > 0$ and let $a, b \in C_F$ satisfy $F(a) \le \epsilon/2$ and $1 - F(b) \le \epsilon/2$. Subdivide the interval $[a, b]$ by $a =$

$a_0 < a_1 < \cdots < a_m = b$ with each $a_j \in C_F$ and such that $a_j - a_{j-1} \leq \epsilon$ for each $j$. Since $F_n \to_d F$, we can find an $N$ so that

$$\max_{1 \leq j \leq m} |F_n(a_j) - F(a_j)| \leq \epsilon/2.$$

for all $n \geq N$. Now we will show that for all $x \in \mathbb{R}$ and all $n \geq N$

$$F(x - \epsilon) - \epsilon \leq F_n(x) \leq F(x + \epsilon) + \epsilon). \tag{1}$$

This will be done by considering the following cases: (i) $x \in [a_{j-1}, a_j]$ for some $j$; (ii) $x \leq a = a_0$; (iii) $x \geq b = a_m$. In case (i), we have

$$\begin{aligned} F_n(x) &\leq F_n(a_j) \leq F(a_j) + \epsilon/2 \leq F(x + \epsilon) + \epsilon/2, \quad \text{and} \\ F_n(x) &\geq F_n(a_{j-1}) \geq F(a_{j-1}) - \epsilon/2 \geq F(x - \epsilon) - \epsilon/2. \end{aligned}$$

In case (ii) (with $x \leq a_0 = a$) we have

$$\begin{aligned} F_n(x) &\leq F_n(a_0) \leq F(a_0) + \epsilon/2 \leq \epsilon \leq F(x) + \epsilon, \quad \text{and} \\ F_n(x) &\geq 0 \geq F(a_0) - \epsilon/2 \geq F(x) - \epsilon/2. \end{aligned}$$

In case (iii) (with $x \geq a_m = b$) we have

$$\begin{aligned} F_n(x) &\leq 1 \leq F(a_m) + \epsilon/2 \leq F(x) + \epsilon/2, \quad \text{and} \\ F_n(x) &\geq F_n(a_m) \geq F(a_m) - \epsilon/2 \geq 1 - \epsilon \geq F(x) - \epsilon. \end{aligned}$$

Thus (1) holds. Since $\epsilon > 0$ is arbitrary, this implies that $\lambda(F_n, F) \to 0$.
(b) Suppose that $\lambda(F_n, F) \to 0$. Let $x_0$ be a continuity point of $F$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|F(x) - F(x_0)| < \epsilon \qquad \text{if} \qquad |x - x_0| \leq \delta.$$

Let $\gamma = \min\{\epsilon, \delta\}$ and let $n$ be so large that $\lambda(F_n, F) < \gamma$. Then

$$\begin{aligned} F_n(x_0) &\geq F(x_0 - \gamma) - \gamma \geq F(x_0) - 2\epsilon, \quad \text{and} \\ F_n(x_0) &\leq F(x_0 + \gamma) + \gamma \leq F(x_0) + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this implies that $F_n(x_0) \to F(x_0)$ for any $x_0 \in C_F$, and hence that $F_n \to_d F$.

- Now we show that the space of one dimensional distribution functions with the metric $\lambda$ is complete.

Suppose that $\{F_k\}$ is a sequence of distribution functions satisfying $\lambda(F_k, F_m) \to 0$ as $k, m \to \infty$. Let $C$ be a countable dense set of points on the real line. Since the values of $F_k(x_j)$ are bounded, the standard diagonal argument proves the existence of a subsequence $F_{k_1}(x),\ F_{k_2}(x),\ \ldots$ which converges for every $x \in C$. The limit

$$G(x_j) = \lim_{r\to\infty} F_{n_r}(x_j)$$

is defined on the set $C$ and is a nondecreasing function. If we define

$$F(x) \equiv \inf_{x_j > x} G(x_j),$$

then $F$ is defined for all $x \in \mathbb{R}$ and is right-continuous. From $\lambda(F_k, F_m) \to 0$ it follows that $F(-\infty) = 0$ and $F(+\infty) = 1$: for every $\epsilon > 0$ there exists a $k$ such that $\lambda(F_k, F_m) < \epsilon$ for $m \geq k$. Further there exists a $z$ such that $F_k(z) < \epsilon$. Then for $x_j < z - \epsilon$,

$$F_{k_r}(x_j) \leq F_n(z) + \epsilon \leq 2\epsilon \qquad \text{for} \ \ k_r \geq n$$

and therefore

$$G(x_j) \leq 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $F(-\infty) = 0$. A similar argument shows that $F(+\infty) = 1$. Now note that $F_{k_r}(x) \to F(x)$ at every continuity point of $F$. Thus it follows that $\lambda(F_{k_r}, F) \to 0$. This together with the hypothesis implies that $\lambda(F_k, F) \to 0$.

- Now we show that the space of one dimensional distribution functions with the metric $\lambda$ is separable.

Let $n \geq 1$ and fix a distribution function $F$. Define a distribution function $F_n$ by

$$\begin{aligned} F_n(x) \ &= \sum_{k=-m2^n+1}^{m2^n} \frac{\lfloor 2^n F(k/2^n) \rfloor}{2^n} 1\left\{ \frac{k-1}{2^n} \leq x < \frac{k}{2^n} \right\} \\ &\quad + \frac{\lfloor 2^n F(-m) \rfloor}{2^n} \cdot 1_{[x<-m]} + \frac{\lfloor 2^n F(m) \rfloor}{2^n} 1_{[x \geq m]} \end{aligned}$$

where $m = m_{n,F}$ is chosen so that $F(-m) \vee (1 - F(m)) \leq 2^{-n}$. Then $F_n \in \mathcal{F}_{n,m}$, the class of all distribution functions concentrated at integer multiplies $k$ of $2^{-n}$ with $-m2^n < k \le m2^m$, and with masses which are also integer multiples of $2^{-n}$. This is a finite set of distribution functions, and $\cup_{m=1}^{\infty} \cup_{n=1}^{\infty} F_{n,m}$ is countable. Moreover for $(k-1)/n \le x < k/n$ we have

$$F_n(x) \le F(k/2^n) = F\left(\frac{k-1}{2^n} + \frac{1}{2^n}\right) \le F(x + 2^{-n}) \le F(x + 2^{-n}) + 2^{-n},$$

while on the other hand

$$F_n(x) \ge \frac{2^n F(k/2^n) - 1}{2^n} = F(k/2^n) - 2^{-n} \ge F(x) - 2^{-n} \ge F(x - 2^{-n}) - 2^{-n}.$$

For $x \ge m$ we have

$$F_n(x) \le F(m) \le F(x) \le F(x + 2^{-n}) + 2^{-n},$$

and on the other hand

$$F_n(x) \ge F(m) - 2^{-n} \ge 1 - 2^{-n} - \cdot 2^{-n} \ge F(x) - 2 \cdot 2^{-n} \ge F(x - 2 \cdot 2^{-n}) - 2 \cdot 2^{-n}.$$

Similarly, for $x \le -m$

$$F_n(x) \le F(-m) \le 2^{-n} \le F(x + 2^{-n}) + 2^{-n}, \qquad \text{and}$$
$$F_n(x) \ge F(-m) - 2^{-n} \ge 0 - 2^{-n} \ge F(x) - 2 \cdot 2^{-n} \ge F(x - 2^{-n}) - 2 \cdot 2^{-n}.$$

Thus we conclude that $\lambda(F_n, F) \le 2 \cdot 2^{-n} \to 0$ as $n \to \infty$. Thus the countable collection $\mathcal{F}_\infty = \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \mathcal{F}_{n,m}$ is dense in $\mathcal{F}$.

5. Exercise 11.6.6, page 34, Wellner, Chapter 11 notes. Formulate and prove an extension of Proposition 2.1 to $\mathbb{R}^k$.

   **Solution:** *Proposition 2.1 in* $\mathbb{R}^k$*:* Suppose that $\{X_n\}$, $X$ are random vectors in $\mathbb{R}^k$ and suppose that $Ef(X_n) \to Ef(X)$ for each $f \in C^\infty(\mathbb{R})$, the class of all bounded functions on $\mathbb{R}^k$ with bounded derivatives of all orders. Then $X_n \to_d X$.

   **Proof.** Let $X \sim N_k(0, I)$. For a fixed $f \in BL(\mathbb{R}^k)$ and $\sigma > 0$ , define a smoothed function $f_\sigma : \mathbb{R}^k \to \mathbb{R}$ as follows:

$$f_\sigma(x) = Ef(x + \sigma Z) = \frac{1}{(2\pi\sigma^2)^{k/2}} \int \cdots \int f(y) \exp\left(-\frac{|x-y|^2}{2\sigma^2}\right) dy.$$

Then $f_\sigma \in C^\infty(\mathbb{R}^k)$ since we can justify repeated differentiation by the dominated convergence theorem. Furthermore

$$\begin{aligned} \sup_{x\in\mathbb{R}^k} |f_\sigma(x) - f(x)| &\le \sup_{x\in\mathbb{R}^k} E|f(x+\sigma Z) - f(x)| \\ &\le \|f\|_{BL} E\{1 \wedge \sigma|Z|\} \end{aligned}$$

where the right side converges to zero as $\sigma \to 0$ by the dominated convergence theorem.

As in the case of $\mathbb{R}$, for fixed $\epsilon > 0$ we can choose $\sigma > 0$ so that $\|f_\sigma - f\|_\infty \le \epsilon$. Then

$$|Ef(X_n) - Ef(X)| \le |Ef_\sigma(X_n) - Ef_\sigma(X)| + 2\epsilon$$

so that

$$\limsup_{n\to\infty} |Ef(X_n) - Ef(X)| \le 2\epsilon$$

since $f_\sigma \in C^\infty(\mathbb{R}^k)$ and hence $Ef_\sigma(X_n) \to Ef_\sigma(X)$ by our hypothesis. Thus $X_n \to_d X$ by the portmanteau theorem.