

Statistics 522, Problem Set 2 Solutions

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1. (Symmetry and conditional expectation). Let X_1, X_2, \dots be i.i.d. random variables with the same distribution as X where $E|X| < \infty$. Let $S_n \equiv X_1 + \dots + X_n$, and define

$$\mathcal{G}_n \equiv \sigma [S_n, S_{n+1}, \dots] = \sigma [S_n, X_{n+1}, X_{n+2}, \dots].$$

Show that $E(X_1|\mathcal{G}_n) = E(X_1|S_n) = n^{-1}S_n$ almost surely. [Hint: Note that $\sigma [X_{n+1}, X_{n+2}, \dots]$ is independent of $\sigma [X_1, S_n]$, and use symmetry to show that $E(1_{[S_n \in B]}X_1) = E(1_{[S_n \in B]}X_2) = \dots = E(1_{[S_n \in B]}X_n)$.]

Solution: First,

$$\begin{aligned} E(X_1|\mathcal{G}_n) &= E(X_1|\sigma [S_n, X_{n+1}, X_{n+2}, \dots]) \\ &= E(X_1|\sigma [S_n]) = E(X_1|S_n) \end{aligned}$$

by (23) of Theorem 8.4.1 using independence of $\sigma [X_1, S_n]$ and $\sigma [X_{n+1}, X_{n+2}, \dots]$. Then note that

$$\begin{aligned} E(X_1 1_{[S_n \in B]}) &= E(X_1 1_{[X_1 + \dots + X_n \in B]}) \\ &= \int \dots \int_{[\sum_1^n x_i \in B]} x_1 dF(x_1) \dots dF(x_n) \\ &= \int \dots \int_{[\sum_1^n x_i \in B]} x_2 dF(x_1) \dots dF(x_n) \\ &= \dots = \int \dots \int_{[\sum_1^n x_i \in B]} x_n dF(x_1) \dots dF(x_n) \end{aligned}$$

where we have used the symmetry of the joint distribution and the fact that the sum $\sum_1^n x_i$ is invariant under relabeling of the coordinates. Thus $E(X_j|S_n) = E(X_1|S_n)$ almost surely for $j = 1, \dots, n$, and it follows that

$$\begin{aligned} n^{-1}S_n &= E(n^{-1}S_n|S_n) \text{ almost surely} \\ &= n^{-1} \sum_{j=1}^n E(X_j|S_n) \\ &= n^{-1}nE(X_1|S_n) \text{ almost surely} \\ &= E(X_1|S_n). \end{aligned}$$

2. (An application of the WLLN: Inversion of Laplace transforms) Let P be a probability measure on the Borel subsets of $[0, \infty)$, and define its Laplace transform by $\varphi(t) = \int_0^\infty e^{-tx} dP(x)$ for $t \in [0, \infty)$. Widder's inversion formula for P from φ is:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = P([0, z]) \quad (1)$$

Show that (1) holds via the following steps:

- (a) Differentiation of the integral k times shows that

$$\varphi^{(k)}(t) = \int_0^\infty (-x)^k e^{-tx} dP(x).$$

- (b) Setting $t = n$, letting $z > 0$, multiplying across by $(-1)^k n^k / k!$, and summing on k yields

$$\sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) = \int_0^\infty \sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} dP(x) \quad (2)$$

where $e^{-nx} (nx)^k / k! = P(S_n = k)$ and $S_n = Y_1 + \dots + Y_n$ where Y_1, Y_2, \dots are i.i.d. $\text{Poisson}(x)$.

- (c) Use the weak law of large numbers and (2) to show that (1) holds.

Solution: (a) and (b) are self-explanatory and follow immediately. It remains only to show that the limit in (c) holds. Now as noted $e^{-nx} \frac{(nx)^k}{k!} = P(S_n = k)$ and $S_n = Y_1 + \dots + Y_n$ where Y_1, Y_2, \dots are i.i.d. $\text{Poisson}(x)$. Hence

$$\begin{aligned} \sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} &= P(S_n \leq [nz]) = P(n^{-1} S_n \leq n^{-1} [nz]) \\ &= P(n^{-1} S_n - n^{-1} [nz] + z \leq z) \\ &= E1\{R_n \leq z\} = Eh_z(R_n) \end{aligned}$$

where the function $h_z(y) \equiv 1\{y \leq z\}$ is bounded and continuous except at the point $y = z$, and where, by the SLLN, $n^{-1}S_n \rightarrow_{a.s.} E(Y_1) = x$ and $n^{-1}[nz] \rightarrow z$ so that $R_n \equiv n^{-1}S_n + z - n^{-1}[nz] \rightarrow_{a.s.} x$ and hence also $R_n \rightarrow_d x$. By the Helly-Bray Theorem 5.1 it follows that

$$\sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} = Eh_z(R_n) \rightarrow h_z(x) = 1\{x \leq z\}$$

for $x \neq z$. Hence for $z \in [0, \infty)$ with $P(\{z\}) = 0$ we have

$$\begin{aligned} \sum_{k=0}^{[nz]} \frac{(-1)^k}{k!} n^k \varphi^{(k)}(n) &= \int_0^\infty \sum_{k=0}^{[nz]} e^{-nx} \frac{(nx)^k}{k!} dP(x) \\ &= \int_0^\infty Eh_z(R_n) dP(x) \\ &\rightarrow \int_0^\infty h_z(x) dP(x) \\ &= \int_0^\infty 1\{x \leq z\} dP(x) = P([0, z]) \\ &= P([0, z)) \end{aligned}$$

if $P(\{z\}) = 0$.

3. PfS, Exercise 8.6.1, page 188. Show that $Z = E(Y|\mathcal{D})$ minimizes $E(Y - Z)^2$ among all random variables $Z \in \mathcal{H}_{\mathcal{D}}$ for each $Y \in \mathcal{H}$.

Solution: Let $Y \in \mathcal{H}$ and set $Z_0 \equiv E(Y|\mathcal{D})$. Then we can write

$$\begin{aligned} E(Y - Z)^2 &= E(Y - Z_0 + Z_0 - Z)^2 \\ &= E(Y - Z_0)^2 + 2E(Y - Z_0)(Z_0 - Z) + E(Z_0 - Z)^2 \\ &= E(Y - Z_0)^2 + 0 + E(Z_0 - Z)^2 \\ &= E(Y - Z_0)^2 + E(Z_0 - Z)^2 \end{aligned}$$

since

$$\begin{aligned} E(Y - Z_0)(Z_0 - Z) &= E\{E\{(Y - Z_0)(Z_0 - Z)|\mathcal{D}\}\} \\ &= E\{(Z_0 - Z)E\{(Y - Z_0)|\mathcal{D}\}\} \quad \text{by (20) of Theorem 8.4.1} \\ &= E\{(Z_0 - Z)\{E(Y|\mathcal{D}) - Z_0\}\} \quad \text{by (14) of Theorem 8.4.1} \\ &= E\{(Z_0 - Z)\{0\}\} \quad \text{by definition of } Z_0 \\ &= 0. \end{aligned}$$

It follows that

$$E(Y - Z)^2 \geq E(Y - Z_0)^2$$

for all $Z \in \mathcal{H}_{\mathcal{D}}$ with equality if and only if $Z = E(Y|\mathcal{D})$ almost surely.

4. PfS, Exercise 8.6.2, page 189: show that $\mathcal{H}_1^0 \perp \mathcal{H}_2^0$ if and only if X_1 and X_2 are independent.

Solution: If X_1 and X_2 are independent, and $Y_j = g_j(X_j)$ for $j = 1, 2$ where $Eg_j(X_j) = 0$ for $j = 1, 2$, then

$$Eg_1(X_1)g_2(X_2) = Eg_1(X_1)Eg_2(X_2) = 0 \cdot 0 = 0, \quad (3)$$

This yields $\mathcal{H}_1^0 \perp \mathcal{H}_2^0$. On the other hand if $\mathcal{H}_1^0 \perp \mathcal{H}_2^0$, then for any $g_j(X_j) \in \mathcal{H}_j^0$ for $j = 1, 2$ we know that (3) holds. Let B_j be Borel sets for $j = 1, 2$. Then take $g_j(x_j) = 1_{B_j}(x_j) - P(X_j \in B_j)$ for $j = 1, 2$. Since $g_j(X_j) \in \mathcal{H}_j^0$ for $j = 1, 2$ it follows that

$$\begin{aligned} 0 &= E(1_{B_1}(X_1) - P(X_1 \in B_1))(1_{B_2}(X_2) - P(X_2 \in B_2)) \\ &= P(X_1 \in B_1, X_2 \in B_2) - P(X_1 \in B_1)P(X_2 \in B_2), \end{aligned}$$

or, equivalently

$$P(X_1 \in B_1, X_2 \in B_2) = P(X_1 \in B_1)P(X_2 \in B_2)$$

for all Borel sets B_1, B_2 . But this implies that X_1 and X_2 are independent random variables.

5. Suppose that $X, Y \in L_1(\Omega, \mathcal{F}, P)$ and that $E(Y|X) = X$ a.s. and $E(X|Y) = Y$ a.s. Prove that $P(X = Y) = 1$.

Solution: (See e.g. Exercise 9.2, Williams, *Probability with Martingales*, page 231.) Suppose first that $X, Y \in L_2(P)$. Then, by Pythagoras (i.e. the orthogonality proved in the solution of problem 3),

$$EX^2 = E(E(X|Y)^2) + E((X - E(X|Y))^2),$$

and since $E(X|Y) = Y$ a.s. this yields

$$E(X^2) = E(Y^2) + E((X - Y)^2). \quad (4)$$

Reversing the roles of X and Y , we also obtain, upon using $E(Y|X) = X$ a.s.,

$$E(Y^2) = E(X^2) + E((Y - X)^2). \quad (5)$$

Adding (4) and (5) gives

$$E(X^2) + E(Y^2) = E(X^2) + E(Y^2) + 2E((X - Y)^2),$$

and this implies that $E(X - Y)^2 = 0$, which in turn implies $P(X = Y) = 1$.

Now one way to proceed is to reduce the general case of $X, Y \in L_1(P)$ to the $L_2(P)$ case treated above. Instead I will prove it using the hint.

Note that

$$\begin{aligned} & E(X - Y)1_{[X > c, Y \leq c]} + E(X - Y)1_{[X \leq c, Y \leq c]} \\ &= E(X - Y)1_{[Y \leq c]} = E(X1_{[Y \leq c]}) - E(Y1_{[Y \leq c]}) \\ &= E(E(X1_{[Y \leq c]}|Y)) - E(Y1_{[Y \leq c]}) \\ &= E(1_{[Y \leq c]}E(X|Y)) - E(Y1_{[Y \leq c]}) \\ &= E(1_{[Y \leq c]}Y) - E(Y1_{[Y \leq c]}) = 0 \end{aligned} \quad (6)$$

using $E(X|Y) = Y$ a.s. in the last line. Similarly, reversing the roles of X and Y ,

$$\begin{aligned} & E(Y - X)1_{[Y > c, X \leq c]} + E(Y - X)1_{[Y \leq c, X \leq c]} \\ &= E(Y - X)1_{[X \leq c]} = 0. \end{aligned} \quad (7)$$

Adding (6) and (7) yields

$$\begin{aligned} 0 &= E(X - Y)1_{[X > c, Y \leq c]} + E(X - Y)1_{[X \leq c, Y \leq c]} \\ &\quad - E(X - Y)1_{[Y > c, X \leq c]} - E(X - Y)1_{[Y \leq c, X \leq c]} \\ &= E(X - Y)1_{[X > c, Y \leq c]} - E(X - Y)1_{[Y > c, X \leq c]}. \end{aligned}$$

Since $[X - Y > 0] = [X > Y] = \cup_{q \in \mathbf{Q}} [X > q \geq Y]$ and similarly for $[X - Y < 0]$, this yields, by summing over rationals q ,

$$0 = E(X - Y)1_{[X - Y > 0]} - E(X - Y)1_{[X - Y < 0]} = E|X - Y|.$$

But this implies $P(|X - Y| = 0) = 1$, or $P(X = Y) = 1$.