Statistics 522, Midterm Exam Solutions

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1. (24 points). **Define** *three* of the following four terms:
   (a) The conditional expectation of a random variable X given a (sub-) sigma-field $\mathcal{D}$.
   (b) Convergence in distribution of a sequence of measures $\{P_n\}$ on a metric space (M, d) to a measure P.
   (c) The class of bounded Lipschitz functions $BL(M)$ on a metric space (M, d).
   (d) The Lévy distance $\lambda(F, G)$ between two distribution functions F and G on $\mathbb{R}$.

   Solution: See PfS chapters 8 and 11 and JAW's chapter 11 notes.

2. (24 points). Give careful **statements** of *any two* of the following four theorems or results:
   (a) The law of the iterated logarithm for i.i.d. random variables $X_1, X_2, \ldots$ with $E(X_1) = 0$ and $Var(X_1) = \sigma^2$.
   (b) The Lindeberg-Feller central limit theorem for a triangular array of row-wise independent random variables with $E(X_{n,i}) = 0$ and $Var(X_{n,i}) = \sigma_{n,i}^2 < \infty$, $i = 1, \ldots, n$.
   (c) The interpretation of conditional expectation $E(Y|\mathcal{D})$ for $Y \in L_2(P)$ as an (orthogonal) projection onto $L_2(\Omega, \mathcal{D}, P)$ where $\mathcal{D} \subset \mathcal{A}$.
   (d) Any CLT result with a rate derived from Stein's method.

   Solution: See PfS chapters 8 and 11 and JAW's chapter 11 notes.

3. (24 points). Let X and Y be two random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ with $E(Y^2) < \infty$. Define $Var(Y|X) = E\{(Y - E(Y|X))^2|X\}$. Show that

$$Var(Y) = EVar(Y|X) + Var(E(Y|X)).$$

Solution: By adding and subtracting $E(Y|X)$ we see that

$$\begin{aligned} Var(Y) &= E(Y - EY)^2 = E(Y - E(Y|X) + E(Y|X) - EY)^2 \\ &= E(Y - E(Y|X))^2 + E\left[(E(Y|X) - EY)^2\right] + 2E\left[(Y - E(Y|X))(E(Y|X) - EY)\right] \\ &= E\left\{E\left[(Y - E(Y|X))^2|X\right]\right\} + E\left[(E(Y|X) - EY)^2\right] + 0 \\ &= EVar(Y|X) + Var[E(Y|X)] \end{aligned}$$

since, by computing conditionally on X and then using linearity of conditional expectation,

$$\begin{aligned} E\left[(Y - E(Y|X))(E(Y|X) - EY)\right] &= E\{E\left[(Y - E(Y|X))(E(Y|X) - EY)|X\right]\} \\ &= E\{(E(Y|X) - EY)E\left[Y - E(Y|X)|X\right]\} \\ &= E\{(E(Y|X) - EY)(E(Y|X) - E(Y|X))\} \\ &= E\{(E(Y|X) - EY) \cdot 0\} = 0. \end{aligned}$$

4. (36 points). Suppose that conditionally on Λ the random variable W has a Poisson distribution where $\Lambda \sim G$ on $\mathbb{R}^+$. That is,

$$P(W = k) = EP(W = k|\Lambda) = \int_0^\infty e^{-v}\frac{v^k}{k!}dG(v), \qquad k \in \{0, 1, 2, \ldots\}.$$

(a) What is the equation characterizing the Poisson(λ) distribution upon which the Stein-Chen approximation method is based?

(b) Show that for each fixed $\lambda \in \mathbb{R}$ the following bound is valid:

$$d_{TV}(P_W, P_\lambda) \leq \min\{1, \lambda^{-1/2}\}E|\Lambda - \lambda|.$$

Here P_λ is the Poisson probability measure on $\mathbb{Z}^+$ with parameter λ: for any $A \subset \mathbb{Z}^+$,

$$P_\lambda(A) = \sum_{z \in A} e^{-\lambda}\frac{\lambda^z}{z!}.$$

Hint: Condition on Λ and use the fact that $\|g\|_\infty \leq \min\{1, \lambda^{-1/2}\}$ for any $g = g_{\lambda,A}$ with $A \subset \mathbb{Z}^+$ solving the Stein-Chen equation $\lambda g(z+1) - zg(z) = 1_A(z) - P_\lambda(A)$ for $z \in \mathbb{Z}^+$.

(c) Suppose that G is the Gamma(r, θ) distribution with density

$$g_{r,\theta}(v) = \frac{\theta^r v^{r-1}}{\Gamma(r)} \exp(-\theta v) 1_{(0,\infty)}(v).$$

Compute $E(W)$ and $Var(W)$ in this case by arguing conditionally. Hint: Use problem 3 above and recall that for $\Lambda \sim$ Gamma(r, θ), $E\Lambda = r/\theta$ and $Var(\Lambda) = r/\theta^2$.

(d) In the same special case as in (c), provide an easy further bound for the bound in (b) when $\lambda \equiv E\Lambda$. (Is there a name for the (unconditional) distribution of W in this case?)

Solution: (a) $X \sim$ Poiss(λ) if and only if

$$E_\lambda\{\lambda g(X+1) - Xg(X)\} = 0$$

for all bounded functions $g : \mathbb{Z}^+ \to \mathbb{R}$.

(b) Let $A \subset \mathbb{Z}^+$ and suppose that g satisfies

$$\lambda g(z+1) - zg(z) = 1_A(z) - P_\lambda(A).$$

Then

$$P(W \in A) - P_\lambda(A) = E\{\lambda g(W+1) - Wg(W)\}. \tag{1}$$

where, since $(W|\Lambda) \sim \text{Poiss}(\Lambda)$,

$$E\{\Lambda g(W+1) - Wg(W)|\Lambda\} = 0.$$

Computing conditionally and using the relation in the last display to replace the second term on the right side of (1) yields

$$\begin{aligned} E\{\lambda g(W+1) - Wg(W)\} &= E\{E[\lambda g(W+1) - Wg(W)|\Lambda]\} \\ &= E\{\lambda g(W+1) - \Lambda g(W+1)\} \qquad (2) \\ &= E\{(\lambda - \Lambda)g(W+1)\}. \qquad (3) \end{aligned}$$

Combination of this identity with (1) yields

$$\begin{aligned} |P(W \in A) - P_\lambda(A)| &\le E\{|\lambda - \Lambda| g(W+1)\} \le \|g\|_\infty E|\Lambda - \lambda| \\ &\le (1 \wedge \lambda^{-1/2}) E|\Lambda - \lambda|. \end{aligned}$$

Hence, by taking the supremum over $A \subset \mathbb{Z}^+$,

$$d_{TV}(P_W, P_\lambda) \le (1 \wedge \lambda^{-1/2}) E|\Lambda - \lambda|.$$

(c) When $\Lambda \sim \text{Gamma}(r, \theta)$, then

$$\begin{aligned} EW &= E\{E(W|\Lambda)\} = E\Lambda = \frac{r}{\theta}, \\ Var(W) &= EVar(W|\Lambda) + Var(E(W|\Lambda)) \\ &= E\Lambda + Var(\Lambda) \\ &= \frac{r}{\theta} + \frac{r}{\theta^2} \\ &= \frac{r}{\theta}\left(1 + \frac{1}{\theta}\right) \\ &> \frac{r}{\theta} = E(W). \end{aligned}$$

(d) When $\Lambda \sim \text{Gamma}(r, \theta)$, and taking $\lambda = E\Lambda = r/\theta$, we find

$$E|\Lambda - E\Lambda| \le \sqrt{Var(\Lambda)} = \frac{\sqrt{r}}{\theta},$$

so the bound of (b) becomes

$$(1 \wedge (r/\theta)^{-1/2}) \cdot \frac{\sqrt{r}}{\theta} = \frac{\sqrt{r}}{\theta} \wedge \frac{1}{\sqrt{\theta}}.$$

In this case W has a (generalized) negative binomial distribution:

$$\begin{aligned} P(W = k) &= \int_0^\infty e^{-v} \frac{v^k}{k!} dG(v) = \frac{1}{k!} \int_0^\infty \frac{v^k \theta^r v^{r-1}}{\Gamma(r)} e^{-v} e^{-\theta v} dv \\ &= \frac{\theta^r}{k!\Gamma(r)(1+\theta)^{k+r}} \int_0^\infty (1+\theta)^{k+r} v^{k+r-1} e^{-(1+\theta)v} dv \\ &= \left(\frac{\theta}{1+\theta}\right)^r \left(\frac{1}{1+\theta}\right)^k \frac{\Gamma(k+r)}{k!\Gamma(r)} \end{aligned}$$

for $k = 0, 1, 2, \ldots$.

Postscript: Another nice bound for the total variation distance of P_W the distribution of a mixture of Poissons to Poisson is given by

$$d_{TV}(P_W, P_{E(\Lambda)}) \le \lambda^{-1}(1 - e^{-\lambda}) Var(\Lambda);$$

see e.g. Barbour, Holst, and Janson (1992), pages 12- 13. When specialized to the case of $\Lambda \sim \text{Gamma}(r, \theta)$, this yields

$$d_{TV}(P_W, P_{r/\theta}) \le (1 \wedge \theta/r)\frac{r}{\theta^2} = \frac{r}{\theta^2} \ \wedge \ \frac{1}{\theta}.$$

5. (35 points). Suppose that $X_n \sim \text{Poisson}(n)$; thus $X_n \stackrel{d}{=} \sum_1^n Y_i$ where Y_i are independent Poisson(1) random variables.
(a) Compute

$$E\left\{\left(\frac{X_n - n}{\sqrt{n}}\right)^-\right\}$$

explicitly and thereby show that this expectation equals $n^{n+1}e^{-n}/(\sqrt{n}n!)$. (Recall that $Y^- = -Y1_{[Y\le 0]}$.)
(b) Show that $(X_n - n)/\sqrt{n} \to_d Z \sim N(0,1)$.
(You may appeal to one of our theorems.)
(c) Use (b) and a uniform integrability argument to show that $E[(X_n - n)^-/\sqrt{n}] \to E[Z^-]$.
(d) Compute $E[Z^-]$.
(e) Combine (a) - (d) to show that $n! \sim \sqrt{2\pi n}(n/e)^n$; i.e. $n!/(\sqrt{2\pi n}(n/e)^n) \to 1$.

Solution: (a) Since $X_n \sim \text{Poisson}(n)$,

$$\begin{aligned} E\left\{\left(\frac{X_n - n}{\sqrt{n}}\right)^-\right\} &= \sum_{k=0}^n \frac{n-k}{\sqrt{n}} e^{-n}\frac{n^k}{k!} \\ &= \frac{e^{-n}}{\sqrt{n}}\left\{\frac{n}{0!} + \sum_{k=1}^n \left(\frac{n^{k+1}}{k!} - \frac{n^k}{(k-1)!}\right)\right\} \\ &= \frac{e^{-n}n^{n+1}}{\sqrt{n}n!} \end{aligned}$$

upon noting that the sum telescopes to just the last term.
(b) Now $n^{-1/2}(X_n - n) = \sqrt{n}(\overline{Y}_n - 1) \to_d Z \sim N(0,1)$ by the CLT.
(c) Since $Z_n \equiv n^{-1/2}(X_n - n) \to_d Z$, it follows by continuous mapping that $Z_n^- \to_d Z^-$. Since $Z_n^- \le |Z_n|$ where $E(Z_n^2) = n^{-1}Var(X_n) = 1$, it follows that $E(Z_n^-) \to E(Z^-)$.
(d) Now

$$E(Z^-) = E(Z^+) = (2\pi)^{-1/2}\int_0^\infty z\phi(z)dz = (2\pi)^{-1/2}.$$

Combining (a) - (d) yields

$$\frac{n^{n+1}e^{-n}}{\sqrt{n}n!} \to \frac{1}{\sqrt{2\pi}}\,.$$

This yields

$$\frac{n!}{\sqrt{2\pi n}(n/e)^n} \to 1\,.$$