Statistics 522, Problem Set 7

Wellner; 2/27/2008

Reading:

Shorack, PfS Sections 12.1-12.3, pages 319 - 329.

Shorack, PfS Sections 13.1 - 13.7, pages 367-390.

Due: Wednesday, March 5, 2008.

1. Graph the first few g_{nj}'s and h_{nj}'s introduced in the handout on the Haar function construction of Brownian bridge and Brownian motion.

2. Justify the interchange of expectation and summation used in the proof of the Theorem in the handout.

3. Let $\mathbb{U}$ be a Brownian bridge process on $[0,1]$. For $0 \le t < \infty$ define a process $\mathbb{B}$ by
$$\mathbb{B}(t) \equiv (1+t)\mathbb{U}\left(\frac{t}{1+t}\right).$$
Show that $\mathbb{B}$ is a Brownian motion process on $[0,\infty)$.

4. PfS, Exercise 3.4, page 328: let $Z_0, Z_1, \ldots$ be i.i.d. $N(0,1)$. Let $f_j(t) \equiv \sqrt{2}\sin(j\pi t)$ for $j \ge 1$; these are orthogonal functions.
(a) Graph the first few f_j's.
(b) Verify that
$$\mathbb{U}(t) \equiv \sum_{j=1}^{\infty} Z_j f_j(t)/j\pi, \qquad 0 \le t \le 1$$
is a Brownian bridge process on $[0,1]$.
(c) Show that with $\mathbb{U}$ as defined in (b) we have
$$W^2 \equiv \int_0^1 \mathbb{U}^2(t)dt = \sum_{j=1}^{\infty} \frac{Z_j^2}{\pi^2 j^2}.$$
(This random variable gives the asymptotic null distribution of the Cramér - von Mises statistic.

5. PfS, Exercise 3.1, page 327.

6. **Optional bonus problem:** Consider the following functional of the partial sum process $\mathbb{S}_n$ treated in Example 5.2, page 30 of (Wellner's) chapter 11:

$$g(x) = \int_0^1 1_{(0,\infty)}(x(t))dt = \lambda\{t \in [0,1] : \ x(t) > 0\} \qquad \text{for } x \in C[0,1]$$

where λ denotes Lebesgue measure.
(a) Show that g is continuous almost surely with respect to the distribution of $\mathbb{S}$ on $C[0,1]$ where $\mathbb{S}$ denotes a Brownian motion process on $C[0,1]$.
(b) Now show that $g(\mathbb{S}_n) \rightarrow_d g(\mathbb{S})$.
(c) What is the distribution of $g(\mathbb{S})$?
(d) What happens if we repeat this for the uniform empirical process treated in Example 5.3, page 32? Does the convergence $g(\mathbb{U}_n) \rightarrow_d g(\mathbb{U})$ hold? What is the distribution of $g(\mathbb{U})$?