Statistics 522, Problem Set 6

Wellner; 2/22/2008

Reading:

Wellner	Chapter 11, Sections 3 - 5, pages 19 - 33.
Shorack, PfS	Section 11.7, pages 312 - 318.

Due: Wednesday, February 27, 2008.

1. Suppose that $\{N_t : t \geq 0\}$ is a Poisson process with rate $\nu > 0$; recall PfS, Chapter 9, pages 195-196 (or, for example, Billingsley, pages 307-313).

   (a) Fix $0 < s < t$, and let $A = [N_s = 0]$, $B_i = [N_t = i]$, $i = 0, 1, 2, \ldots$, and let $\mathcal{D} \equiv \sigma[B_1, B_2, \ldots]$. Show that $P(N_s = 0 | \mathcal{D}) = (1 - s/t)^{N_t}$.

   (b) Fix $0 < s < t$. Show that

$$P(N_s = k | N_t) = \binom{N_t}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{N_t - k} 1_{\{0,\ldots,N_t\}}(k).$$

   Thus $(N_s | N_t) \sim \text{Binomial}(N_t, s/t)$.

   (c) Fix $0 \equiv s_0 < s_1 < \cdots < s_{m-1} < s_m \equiv t$. Show that

$$\begin{aligned} &P(\cap_{j=1}^m \left[N_{s_j} - N_{s_{j-1}} = k_j\right] | N_t) \\ &\quad = \frac{N_t!}{k_1! \cdots k_m!} \prod_{j=1}^m \left(\frac{s_j - s_{j-1}}{t}\right)^{k_j} 1_{\{0 \le k_j \le N_t,\ j=1,\ldots,m,\ \sum_{j=1}^m k_j = N_t\}}. \end{aligned}$$

   That is,

$$\begin{aligned} &(N_{s_1}, N_{s_2} - N_{s_1}, \ldots, N_t - N_{s_{m-1}} \big| N_t) \\ &\qquad \sim \text{Mult}_m \left(N_t, \left(\frac{s_1}{t}, \frac{s_2 - s_1}{t}, \ldots, \frac{t - s_{m-1}}{t}\right)\right). \end{aligned}$$

   Remark. The result of (c) implies that

$$(\{N_s :\ 0 \le s \le t\} | N_t = n) \overset{d}{=} \{n\mathbb{F}_n(s) :\ 0 \le s \le t\}$$

   where $\mathbb{F}_n(s) = n^{-1} \sum_{i=1}^n 1_{[0,s]}(X_i)$ is the empirical distribution function of X_i i.i.d. Uniform$(0, t)$.

2. Suppose that $X_1, \ldots, X_n, \ldots$ are i.i.d. with distribution function F, and let N be a Poisson random variable with parameter λ which is independent of the X_i's. Define the stochastic process $\{M_t : t \in \mathbb{R}\}$ by
$$M_t = \sum_{i=1}^{N} 1_{(-\infty,t]}(X_i) = N\mathbb{F}_N(t)$$
where $\mathbb{F}_n(t) = n^{-1}\sum_{i=1}^{n} 1_{(-\infty,t]}(X_i)$ is the empirical distribution function of $X_1, \ldots, X_n$.
(a) Show that $M_t \sim Poisson(\lambda F(t))$ for each fixed $t \in \mathbb{R}$.
(b) Show that for fixed $-\infty \equiv s_0 < s_1 < s_2 < \cdots < s_{m-1} < s_m \equiv \infty$, the random variables $M_{s_1}, M_{s_2} - M_{s_1}, \ldots, M_{s_m} - M_{s_{m-1}}$ are independent Poisson random variables with parameters $\lambda F(s_1), \lambda(F(s_2) - F(s_1)), \ldots, \lambda(1 - F(s_{m-1}))$ respectively.
Remark: (a) and (b) imply that the process M_t is a Poisson process with intensity (and mean) function λF; this fact generalizes to the empirical measure of observations X_i i.i.d. P on $(\mathcal{X}, \mathcal{A})$; see e.g. van der Vaart and Wellner (1996), section 3.5.2.

3. Suppose that $X_1, \ldots, X_n, \ldots$ are independent and identically distributed with $E|X_1| < \infty$, and let $S_n = X_1 + \cdots + X_n$. Let $\mu = E(X_1)$. Suppose that N is an integer-valued random variable independent of the X_i's.
(a) If $EN < \infty$, show that $E(S_N) = \mu E(N)$.
(b) If $Var(N) < \infty$, and $Var(X_1) = \sigma^2 < \infty$, compute $Var(S_N)$.
Remark: Consider Example 10.5.3, page 245, and Theorem 11.6.1, page 309, in this connection, noting that the random number of terms in these results need *not* be independent of the X_i's being summed.

4. Exercise 11.8.16, page 56, Wellner, Chapter 11, notes.

5. **Optional bonus problem:** Exercise 11.8.11, page 56, Wellner, Chapter 11, notes.

6. **Optional bonus problem:** Exercise 11.8.15, page 56, Wellner, Chapter 11, notes.