Statistics 522, Problem Set 4

Wellner; 2/05/2008

Reading:

Wellner Chapter 11, Sections 3 - 5, pages 19 - 33.

Due: Wednesday, February 13, 2008.

1. Suppose that $Y_1, \ldots, Y_n$ be independent Bernoulli (1/2) random variables so that $X_j = 2Y_j - 1$ are independent Rademacher random variables. with $EX_j = 0$ and $Var(X_j) = 1$. Let $S_n = X_1 + \cdots + X_n$, and $\widetilde{S}_n = S_n/\sqrt{n}$, and define $F_n(x) = P(\widetilde{S}_n \le x)$. Show that
$$\limsup_{n\to\infty} \left\{ \sqrt{n} \|F_n - \Phi\|_\infty \right\} \ge \frac{1}{\sqrt{2\pi}}.$$
Hint: Take $n = 2m+1$; take $t_m = -(2m+1)^{-1/2}$; note that $\sum_1^n Y_j \sim$Binomial$(n, 1/2)$; show that $P(Bin(2m+1, 1/2) \le m) = 1/2$; and hence that $F_{2m+1}(t_m) - \Phi(t_m) = \int_{t_m}^0 \phi(y)dy$.

2. Stroock's chapter II, page 64: verify the claim that the bound in Lindeberg's CLT can be taken to be the expression in (2.1.13) of Stroock when the X_{ni}'s all have finite third moments.

3. (Stein's method for Poisson random variables). Suppose that $X \sim$ Poisson(λ), and write P_λ for the corresponding Poisson probability measure on $\mathbb{Z}^+$ and E_λ for expectation under P_λ. Show that the following identity holds: for any bounded function $g : \mathbb{Z}^+ \mapsto \mathbb{R}$,
$$E_\lambda\{\lambda g(X+1) - X g(X)\} = 0.$$

4. (Stein's method for Poisson random variables, continued). Let $A \subset \mathbb{Z}^+$. Consider the bounded function $\varphi(z) = 1_A(z)$ for $z \in \mathbb{Z}^+$ and its Poisson-centered version $\widetilde{\varphi}(z) = \varphi(z) - E_\lambda \varphi(X)$. Suppose that $g = g_{\lambda,A}$ is constructed to solve the equation
$$\lambda g(z+1) - z g(z) = \widetilde{\varphi}(z) = 1_A(z) - P_\lambda(A). \qquad (1)$$

Show that the solution of the equation in the last display is given by

$$\begin{aligned} g(z+1) &= \lambda^{-z-1} z! e^{\lambda} \{P_\lambda(A \cap U_z) - P_\lambda(A) P_\lambda(U_z)\} \\ &= \lambda^{-z-1} z! e^{\lambda} \{P_\lambda(A \cap U_z) P_\lambda(U_z^c) - P_\lambda(A \cap U_z^c) P_\lambda(U_z)\} \end{aligned}$$

for $z \geq 0$, $z \in \mathbb{Z}^+$ where $U_z \equiv \{0, \ldots, z\}$.

5. (Stein's method for Poisson random variables, continued). Suppose that $W_n = \sum_{i=1}^n X_i$ where $X_1, \ldots, X_n$ are independent, $X_i \sim$ Bernoulli(p_i). Let $\lambda \equiv \lambda_n = E(W_n) = \sum_{i=1}^n p_i$. Use the identity of the previous problem to show that for any set $A \subset \mathbb{Z}^+$

$$|P(W \in A) - P_\lambda(A)| \leq \left\{ \begin{array}{l} 2 \sup_z |g_{\lambda,A}(z)| \sum_{i=1}^n p_i^2 \\ \sup_z |g_{\lambda,A}(z+1) - g_{\lambda,A}(z)| \sum_{i=1}^n p_i^2. \end{array} \right.$$

Hint: Use the identity in the previous problem, then note that $E\{X_i g(W)\} = E\{X_i g(W_i + 1)\} = p_i E g(W_i + 1)$ where $W_i = \sum_{j \neq i} X_j$. Furthermore, note that W and W_i are equal unless $X_i = 1$, which occurs with probability p_i.

6. (Stein's method for Poisson random variables, continued). It is shown in Barbour, Holst, and Janson (1992) that for $g = g_{\lambda,A}$ solving (1), the following bounds hold:

$$\|g\|_\infty \leq \min\{1, \lambda^{-1/2}\}, \qquad \|\Delta g\|_\infty \leq \lambda^{-1}(1 - e^{-\lambda}).$$

Use these bounds to show that

$$d_{TV}(P(W_n \in \cdot), P_\lambda) \leq \lambda^{-1}(1 - e^{-\lambda}) \sum_{i=1}^n p_i^2,$$

which improves substantially on the bound $\sum_{i=1}^n p_i^2$ established in problem 6, Math/Stat 521 homework set 6, via coupling.